OHIO STATE
UNIVERSITY

JOURNAL OF MATHEMATICS AND PHYSICS

Massachusetts Institute of Technology

Volume VII
1927 – 1928

EDITORS

FREDERICK S. WOODS, Professor of Mathematics
HARRY M. GOODWIN, Professor of Physics and Electro-Chemistry
FREDERICK G. KEYES, Professor of Physico-Chemical Research

MANAGING EDITOR

CLARENCE L. E. MOORE, Professor of Mathematics

QA 1
J87
v.7

OHIO STATE

CONTENTS

	PAGE
A Relativistic Theory of Quanta By D. J. STRUIK AND NORBERT WIENER	1
A General Operational Analysis By W. O. PENNELL	24
Multiple Invariants and Generalized Rank of a P-Way Matrix or Tensor By FRANK L. HITCHCOCK	39
Parallel Geodesics By C. L. E. MOORE	81
A Qualitative Definition of the Sub- and Super-Harmonic Functions By PHILIP FRANKLIN	86
Couche Ranks in the General Matrix By LEPINE HALL RICE	93
Analytic and Topological Transformations of Closed Surfaces By EDWARD SAIBEL	97
Coherency Matrices and Quantum Theory By NORBERT WIENER	109
A Method for Deriving Formulas for the Approximate Calculation of Integrals By PRESCOTT D. CROUT	126
A New Method in Tauberian Theorems By NORBERT WIENER	161
Note on Congruences in Riemannian Space By C. L. E. MOORE	185
On Sets of Principal Directions in a Riemannian Manifold of Four Dimensions By D. J. STRUIK	193
Deformation of Plane Pipes By WILLIAM HOVGAARD	198
Further Research on Pipe Bends By WILLIAM HOVGAARD	239
On the Methods of Classical and Quantum Mechanics By N. H. FRANK	298
A New Machine for Integrating a Functional Product By KING E. GOULD	305

A RELATIVISTIC THEORY OF QUANTA

By D. J. Struik and Norbert Wiener

Introduction

It is the purpose of the present paper to develop a form of the theory of relativity which shall contain the theory of quanta, as embodied in Schrödinger's[1] wave mechanics, not merely as an afterthought, but as an essential and intrinsic part. The mathematical theory underlying our work is Cotton's theory of the differential invariants of linear partial differential equations of the second order.[2] We arrive at the possibly startling conclusion that there is an intimate and direct connection between the Einstein field equations and the Fock-De Donder form[3] of the Schrödinger wave equation in general relativity.

The fundamental idea of the Einstein theory is that the laws of physics should be stated in a form invariant under a general point-transformation of space-time, or in other words, in tensor form. In order for such a theory to assume a content, there must be certain fundamental physical tensors of the universe to which it applies. Einstein takes as his fundamental tensor a

[1] See the various memoirs which Schrödinger has collected into his "Abhandlungen zur Wellenmechanik" (Leipzig, J. A. Barth, 1927).

[2] *E. Cotton*, Sur les invariants différentiels de quelques équations linéaires aux dérivées partielles du second ordre. Ann. Ec. Norm. Sup. (3) 17 (1900), pp. 211–244.

Cf. also *T. Levi Civita*, Sulla trasformazione delle equazioni lineari a derivate parziali del secondo ordine. Atti R. Ist. Veneto 3a, ser. 15, tomo 72 (1912–13), pp. 1331–1357.

As we shall point out, these papers contain the potentiality of the entire Weyl theory of linear connection. They have not, however, exercised an influence on the development of relativity and of quantum theory. Recently *Th. De Donder* and *G. van Lerberghe* have published a note in which they develop a portion of Cotton's theory for application to quantum theory (Théorie invariantive des ondes (Première communication) Académie Royale de Belgique, Classe des sciences, 5e série, tome XIII, pp. 191–200). They do not seem, however, to be aware of the existence of Cotton's prior work on the subject.

[3] *V. Fock*. Über die invariante Form der Wellen-und der Bewegungsgleichungen für einen geladenen Massenpunkt. Zs. für Physik 39 (1926), pp. 226–232.

Th. De Donder. Bull. Classe des Sciences, Acad. Roy. de Belgique, séance du 5 février et du 5 mars 1927, comp. also his book "The mathematical theory of relativity," published by Mass. Inst. of Techn., 1927, x+102 p.

337975

second order covariant tensor $g_{\lambda\mu}$, which determines "interval" in the universe in accordance with the definition

$$ds^2 = \sum_{\lambda,\mu}^{1,\cdots,4} g_{\lambda\mu} dx^\lambda dx^\mu. \qquad (0.1)$$

This tensor determines the kinematical-gravitational properties of the universe.

So long as the universe is conceived as an assemblage of particles in motion, the concept of interval is natural. It was at once apparent, however, that electromagnetic phenomena require the introduction of other non-gravitational tensors. These tensors occupy a secondary, extra-geometrical position in the first forms of Einstein's theory. It is the service of Weyl to have introduced a more general form of geometry in which the roles of electricity and of gravity are equally recognized.

Weyl's geometry has been shown to generate a unique Einstein gravitational field and a unique associated electromagnetic vector. This is done through the introduction of the idea of natural gauge.[4] That is, the physical significance of the Weyl theory is not so much the invention of a new type of geometry and physics as the discovery of a new approach to the Einstein physics and the Riemann geometry. In a world of mixed gravitational and electromagnetic phenomena, a particular phenomenon does not have its gravitational and electromagnetic parts separately labelled. Weyl's theory gives us a rule for dissecting the gravitational and the electromagnetic aspects of the world away from one another.

There are, however, two difficulties in the Weyl theory. The first is that the Weyl linear connection is but one among a vast congeries of such connections, and that its preferred position is in need of explanation. If a Weyl physics, why not an Eddington physics, a Schouten physics? The second and by far the more important difficulty is that in addition to gravitational and electromagnetic phenomena, the world contains a third class, that of quantum phenomena, and that the Weyl theory contains no place for these.

Thus the need for a new theory appears: that is, for a method of dividing the mixed gravitational-plus-electromagnetic-plus-

[4] *H. Weyl.* Raum. Zeit, Materie, 4. Aufl. 1921, p. 269, comp. *A. S. Eddington*, The mathematical theory of relativity, 2d ed. (1924), p. 207.

quantum phenomena of the world into their three parts, and of defining the entities with which the Einstein and the Maxwell theories deal. It is natural to seek for a suggestion of the form of such a theory in that modification of the theory of quanta which has recently made the greatest progress: that is, in the Schrödinger wave mechanics. This theory finds that the central phenomenon of what we know as the motion of a single electron in an electromagnetic field of force is a wave motion represented by a certain second order linear homogeneous differential equation of the properly hyperbolic type in four independent variables. The question thus arises, can we proceed from an equation of this type to an unambiguous definition of a gravitational and an electromagnetic field? This question we shall answer in the affirmative. We shall further discuss the effect of the Einstein field equations on the form of the original wave equation. In other words, we shall carry the Weyl theory of natural gauge one step further, and shall thus provide a place for quantum phenomena. We shall indeed explain why it is that just the Weyl type of linear connection is physically important.

It may be interesting to trace the history of the Schrödinger theory. It owes its inception to the brilliant suggestion of de Broglie, that the quantum principle of the selection of certain orbits is due to interference effects from a wave accompanying the motion of an electron, in such a manner that the electron moves along a ray of the wave.[5] Schrödinger's step was to discard the particle-electron altogether, and to account for the selection of certain specific levels of energy by the moving electron, by the theory of the characteristic functions of a linear partial differential equation. The first form of the Schrödinger theory was non-relativistic, but both Schrödinger and de Broglie soon modified it[6] so as to make it include the phenomena of restricted relativity.

The non-relativistic Schrödinger equation could be employed to

[5] *L. de Broglie.* Recherches sur la théorie des quanta. Annales de Physique, 10e série, 3 (1925), pp. 22–128. This paper was preceded by several notes in the Comptes Rendus.

[6] *E. Schrödinger.* Quantisierung als Eigenwertproblem. Vierte Mitteilung. Annalen der Physik (4) Vol. 81 (1926), pp. 109–139. Book referred to sub. (1), pp. 139–169.

L. de Broglie. Les principes de la nouvelle mécanique ondulatoire. Journal de Physique (6) 7 (1926), pp. 321–337.

give an account of the Stark and the Zeeman effects. To this end, Schrödinger developed a form of the theory of pertubations. On the other hand, the so-called fine structure of the lines of the hydrogen atom is a relativistic effect even in the Bohr-Sommerfeld particle mechanics theory. However, when Schrödinger attempted to use his relativistic theory to account for the fine structure, he was met with the difficulty that while his wave theory gave a fine structure, it gave half quantum numbers for the so-called azimuthal number. As there was a certain amount of evidence that the real azimuthal quantum numbers should have half values, which are masked by a phenomenon of the nature of the Goudsmid and Uhlenbeck spinning electron, Schrödinger did not regard this difficulty as fatal. Recently Epstein[7] has carried through in detail the mathematical analysis of the fine structure, using a special case of Schrödinger's relativistic-magnetic generalized equation as his starting point. He supposes that the field of force in which the hydrogen electron moves is not that of a mere electrostatic point-charge, but has superimposed on it a certain small magnetic dipole. He obtains the correct energy levels, but his work has been criticized in so far as the determination of the intensities of the hydrogen lines is concerned. This criticism probably concerns the interpretation of his results more than the differential equation which he chooses. It may be said at any rate that Schrödinger's relativistic-magnetic equation has proved adequate for the discussion of all the quantized orbits of the single charged particle under special relativity.

We have already cited the papers of Klein and De Donder which accomplish the transition to general relativity. Neither of these papers derives the fundamental gravitational tensor from the Schrödinger equation itself, but starts instead with a given metric. Although De Donder and Van Lerberghe devote a later paper to a preliminary discussion of the invariantive theory of the Schrödinger equation, they do not proceed to the stage of our "normalization theory," which is the analogue of the Weyl theory of natural gauge. In this paper, however, the quantum

[7] *P. Epstein.* The magnetic dipole in undulatory mechanics. Proc. Nat. Acad. Sc. U. S. A. 13 (1927), pp. 232–237. Comp. also *C. F. Richter.* The hydrogen atom with a spinning electron in wave mechanics. *Ib.* pp. 476–479.

theory and relativity are both derived from the invariant theory of a single differential equation.

To outline our procedure: we shall reconstruct the essential features of the Cotton invariant theory. We shall develop our theory of normalization, and shall give a physical interpretation of the coefficients of our normalized equation. We shall discuss the meaning of the Einstein field equations, as applied to the Schrödinger equation. We shall discuss the Kaluza-Fock-Klein five-dimensional quantum theory.[8] We shall show that it is merely the homogeneous theory corresponding to our fundamental equation, which is non-homogeneous in order. We shall show that it is equivalent with our theory except in the important respect, that it does not admit of a theory of normalization.

We shall finally make some general remarks concerning the problems of two and more bodies, outlining what seem to us to be promising modes of attack.

Part I. The Mathematical Theory[9]

§1. The Fundamental Form. We consider an expression in n independent variables x^ν, $\nu=1, 2, \ldots, n$ of the form

$$F(\psi) \equiv \sum_{\lambda,\mu}^{1,\ldots,n} a_{\lambda\mu} \frac{\partial^2\psi}{\partial x^\lambda \partial x^\mu} + \sum_{\lambda}^{1,\ldots,n} a_\lambda \frac{\partial\psi}{\partial x^\lambda} + a_0\psi \qquad (1.1)$$

Here $a_{\lambda\mu}$, a_λ and a_0 are functions of the x^ν only. All functions considered in this paper are to be continuous with continuous first and second derivatives over the region over which we consider them.

The given form (1.1) is called the *fundamental form*. If these variables are transformed into another set

$$'x^\nu = f^\nu(x^1, x^2, \ldots, x^n) \qquad (1.2)$$

[8] *Th. Kaluza.* Zur Unitätsproblem der Physik. Sitzungsberichte Berliner Akademie, Physik.-Mathem. Classe 1921, pp. 966–972.
O. Klein. Quantentheorie und fünfdimensionale Relativitätstheorie. Zs. für Phys. 37 (1926), pp. 895–906.
V. Fock, l. c. sub. [3].

[9] In Part I we give a sketch of Cotton's theory of a linear partial differential equation of the second order.

with non-vanishing determinant D:

$$D = \left| \frac{\partial' x^\nu}{\partial x^\lambda} \right| \neq 0, \tag{1.3}$$

the fundamental form is changed into another form of the same construction

$$'F(\psi) \equiv \sum_{\lambda,\mu}^{1,\cdots,n} {}'a_{\lambda\mu} \frac{\partial^2 \psi}{\partial' x^\lambda \partial' x^\mu} + \sum_{\lambda}^{1,\cdots,n} {}'a_\lambda \frac{\partial \psi}{\partial' x^\lambda} + {}'a_0 \psi \tag{1.4}$$

The equations connecting the old and the new coefficients are

$${}'a_{\lambda\mu} = \sum_{\rho\sigma} a_{\rho\sigma} \frac{\partial' x^\lambda}{\partial x^\rho} \frac{\partial' x^\mu}{\partial x^\sigma} \tag{1.5}$$

$${}'a_\lambda = \sum_{\rho\sigma} a_{\rho\sigma} \frac{\partial^2 {}' x^\lambda}{\partial x^\rho \partial x^\sigma} + \sum_{\rho} a_\rho \frac{\partial' x^\lambda}{\partial x^\rho} \tag{1.6}$$

$${}'a_0 = a_0. \tag{1.7}$$

By definition, a contravariant vector is a system of components v^ν which are transformed under the transformations (1.2) according to the rule

$${}'v^\nu = \sum_{\lambda} v^\lambda \frac{\partial' x^\nu}{\partial x^\lambda} \tag{1.8}$$

We thus see from (1.5) that the coefficients $a_{\rho\sigma}$ behave like the components of a contravariant tensor, and in accordance with this, we write $g^{\rho\sigma}$ in place of $a_{\rho\sigma}$. Our law of transformation now reads

$${}'g^{\lambda\mu} = \sum_{\lambda,\mu}^{1,\cdots,n} g^{\rho\sigma} \frac{\partial' x^\lambda}{\partial x^\rho} \frac{\partial' x^\mu}{\partial x^\sigma}; \quad (g^{\lambda\mu} = g^{\mu\lambda}). \tag{1.9}$$

We see from (1.6) that the a_λ do not behave like the components of a contravariant vector, inasmuch as the first term of the right-hand member appears in their equations of transformation. It is, however, not difficult to derive from the a_λ a more complicated expression with vector character, provided that Δ, the determinant of the $g^{\lambda\mu}$, does not vanish. That is,

$$\Delta = |g^{\lambda\mu}| \neq 0. \tag{1.10}$$

Under this hypothesis, there exists a covariant tensor

$$\frac{\text{Minor of } g^{\lambda\mu} \text{ in } \Delta}{\Delta} = g_{\mu\lambda} = g_{\lambda\mu} \tag{1.11}$$

and a covariant differential operator ∇_μ:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} \tag{1.12}$$

$$\nabla_\mu v_\lambda = \frac{\partial v_\lambda}{\partial x^\mu} - \sum_\nu \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} v_\nu,$$

where $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ are the Christoffel symbols of the second order with respect to the $g_{\lambda\mu}$.

Let us now consider the generalized Laplacian of ψ:

$$\begin{aligned} \Box\psi = \sum_{\lambda,\mu} g^{\lambda\mu} \nabla_\lambda \nabla_\mu \psi &= \sum_{\lambda,\mu} g^{\lambda\mu} \frac{\partial^2 \psi}{\partial x^\lambda \partial x^\mu} - \sum_{\lambda,\mu,\nu} g^{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} \frac{\partial \psi}{\partial x^\nu} \\ &= \frac{1}{\sqrt{-g}} \sum_{\lambda,\mu} \frac{\partial}{\partial x^\mu} \sqrt{-g}\, g^{\lambda\mu} \frac{\partial \psi}{\partial x^\lambda}, \end{aligned} \tag{1.13}$$

where $g = \Delta^{-1}$ is the determinant of the $g_{\lambda\mu}$. We see that (1.1) can be written as follows:

$$F(\psi) \equiv \Box\psi + \sum_{\lambda,\mu,\nu} \left[a_\lambda + g^{\mu\nu} \left\{ \begin{matrix} \mu\nu \\ \lambda \end{matrix} \right\} \right] \frac{\partial \psi}{\partial x^\lambda} + a_0 \psi. \tag{1.14}$$

From (1.14) and (1.4) we can deduce the fact that the coefficients of $\frac{\partial \psi}{\partial x^\lambda}$ in this equation have the character of the components of a contravariant vector. We shall call this vector $-2p^\lambda$; so that:

$$p^\lambda = -\frac{1}{2} a_\lambda - \frac{1}{2} \sum_{\mu\nu} g^{\mu\nu} \left\{ \begin{matrix} \mu\nu \\ \lambda \end{matrix} \right\}. \tag{1.15a}$$

We may also write this in the form

$$p^\lambda = -\frac{1}{2} a_\lambda + \frac{1}{4} \sum_\mu g^{\lambda\mu} \frac{\partial \log g}{\partial x^\mu} - \frac{1}{2} \sum_\mu \frac{\partial g^{\lambda\mu}}{\partial x^\mu}. \tag{1.15b}$$

By introducing p^λ in (1.14), we may write our fundamental form in the invariant way

$$F(\psi) \equiv \Box\psi - 2\sum_\lambda p^\lambda \frac{\partial\psi}{\partial x^\lambda} + a_0\psi. \tag{1.16}$$

This gives us

Theorem I. *A fundamental form $F(\psi)$ determines a contravariant tensor $g^{\lambda\mu}$ and a scalar function a_0. If the determinant of the $g^{\lambda\mu}$ does not vanish, it also determines a covariant tensor $g_{\lambda\mu}$ and a contravariant vector p^λ.*

From p^λ we then may derive the covariant vector

$$p_\mu = g_{\lambda\mu}p^\lambda \text{ [10]} \tag{1.17}$$

§2. **The Fundamental Equation.** We now consider the invariantive theory of the equation

$$F(\psi) = 0 \tag{2.1}$$

under those transformations of the form (1.2) which satisfy (1.3). This is equivalent to the conformal theory of the fundamental form. Equation (2.1) is called the *fundamental equation*. We suppose that g does not vanish.

Let us subject the fundamental form to the conformal transformation

$$'F(\psi) = RF(\psi) \tag{2.2}$$

where R is a function of the n variables x^ν. Then

$$\begin{aligned} 'g^{\lambda\mu} &= Rg^{\lambda\mu} \\ 'g_{\lambda\mu} &= R^{-1}g_{\lambda\mu} \\ 'a_0 &= Ra_0 \end{aligned} \tag{2.3}$$

[10] It may be shown to follow from equations (1.6) that in any geometry of paths determined by a set of $\Gamma^\nu_{\lambda\mu}$, where a form (1.1) is introduced, the laws of transformation of $g^{\lambda\mu}\,\Gamma^\nu_{\lambda\mu}$ assume the same form as that which is assumed by our a_ν. If we restrict the $\Gamma^\nu_{\lambda\mu}$ by the equation

$$a_\nu = \sum_{\lambda\mu} g^{\lambda\mu}\,\Gamma^\nu_{\lambda\mu},$$

the fundamental form $F(\Psi)$ may be written

$$F(\Psi) = \sum_{\lambda\mu} g^{\lambda\mu}\,\nabla\,\nabla_\mu\Psi + a_0\Psi,$$

where the ∇_λ represent covariant differentiation with respect to the $\Gamma^\nu_{\lambda\mu}$.

and the determinant of the $g_{\lambda\mu}$ is transformed according to the formula:

$$'g = R^{-n} g. \tag{2.4}$$

As

$$'a_\lambda = R a_\lambda \tag{2.5}$$

and

$$'g^{\lambda\mu}\left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} = R g^{\lambda\mu}\left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} + \frac{1}{2}(n-2) g^{\mu\nu}\frac{\partial R}{\partial x^\nu}, \tag{2.6}$$

we find

$$'p^\nu = R p^\nu - \frac{1}{4}(n-2) g^{\mu\nu}\frac{\partial R}{\partial x^\mu}. \tag{2.7}$$

In (2.6) and (2.7) we follow the now familiar custom, and omit the sign Σ where two identical indices occur in the same term underneath and we shall adhere to this practice in the future.

From (2.7) and (2.3) we obtain

$$'p_\mu = p_\mu + \frac{1}{4}(n-2)\frac{\partial \log R}{\partial x^\mu}. \tag{2.8}$$

That is, p_μ is altered by an additive term which is a gradient function. Now, the linear connection introduced by Weyl[11] is determined by a vector ϕ_μ and a tensor $g_{\lambda\mu}$ which are subject to the following transformations:

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \qquad '\phi_\mu = \phi_\mu - \frac{\partial \log \sigma}{\partial x^\mu}. \tag{2.9}$$

If we put $\sigma = R^{-1}$ and $\phi_\mu = \frac{4}{n-2} p_\mu$, we obtain our formulas (2.3) and (2.8). Hence

Theorem II. *A fundamental equation $F(\psi)=0$ determines in a unique way a linear connection of the Weyl type, with metrical tensor $g_{\lambda\mu}$ and gauge vector $\frac{4}{n-2}\, p_\mu$.*

The last term in $F(\psi)=0$ has no influence on the Weyl connection. Indeed, it may even vanish without affecting this connection.[12]

[11] Cf. *e.g.*, *H. Weyl*, Raum, Zeit, Materie, (Berlin J. Springer) 4e Auflage 1921, p. 260.

[12] Such equations have been considered by *T. Levi Civita*, l. c. sub [2].

We thus see that the scalars and tensors belonging to the Weyl connection are uniquely determined by the fundamental equation. As examples we may mention, if the Weyl Riemann-Christoffel curvature tensor is $G_{\omega\mu\lambda}^{\cdots\cdots\nu}$, the Weyl contracted curvature tensor $G_{\omega\mu\lambda}^{\cdots\cdots\omega}$ and the Weyl curvature scalar G. This latter is, however, not an *absolute* invariant with respect to a conformal transformation. It is defined by the formula

$$\begin{aligned} G &= K+4\frac{n-1}{n-2}(\nabla_\mu p_\lambda)g^{\mu\lambda}-4\frac{n-1}{n-2}p_\mu p_\lambda g^{\mu\lambda} \\ &= K+4\frac{n-1}{n-2}(\nabla_\mu p^\mu - p_\mu p^\mu),^{13} \end{aligned} \tag{2.10}$$

where K is the curvature scalar of the $g_{\lambda\mu}$, and its law of transformation under a conformal transformation is

$$'G = RG. \tag{2.11}$$

It follows from this equation and (2.5) that every linear combination of G and a_0 has the same law of transformation:

$$'G+I'a_0 = R(G+Ia_0). \tag{2.12}$$

Here I can assume the value of any invariant with respect to the transformation in question.

The curl of p_λ is also an absolutely invariant tensor, since we have by (2.8) under transformations (2.2)

$$\frac{\partial' p_\lambda}{\partial' x^\mu}-\frac{\partial' p_\mu}{\partial' x^\lambda}=\frac{\partial p_\lambda}{\partial x^\mu}-\frac{\partial p_\mu}{\partial x^\lambda}. \tag{2.13}$$

This bivector is proportional to the alternating part of the contracted Weyl curvature tensor.[14]

§3. **The Normalization of the Fundamental Equation.** As the solutions of the fundamental equation are not changed by the multiplication of the equation with any function R of the x^ν, we may so determine that the equation assumes a normal form.

[13]Cf. *A. S. Eddington*, The Mathematical Theory of Relativity. Cambridge, University Press, 2d. ed., 1924, p. 205.

[14]Eddington, loc. cit. Cf. *e.g.*, J. A. Schouten, Der Ricci Kalkül, Berlin, J. Springer, 1924, p. 219.

A natural mode of normalization is suggested by equation (2.12). Let us determine R so that $G+Ia_0$ assumes a given value V:

$$R(G+Ia_0)=V \tag{3.1}$$

or

$$R=\frac{V}{G+Ia_0} \tag{3.2}$$

In the normalized equation:

$$K+4\frac{n-1}{n-2}(\nabla_\mu p^\mu-p_\mu p^\mu)+Ia_0=V. \tag{3.3}$$

We may hence write it in the form

$$\Box\psi-2p^\lambda\frac{\partial\psi}{\partial x^\lambda}+\frac{1}{\lambda}\left[V-K+4\frac{n-1}{n-2}(p_\mu p^\mu-\nabla_\mu p^\mu)\right]\psi=0. \tag{3.4}$$

If we put

$$I=4\frac{n-1}{n-2}$$

we get

$$\Box\psi-2p^\lambda\frac{\partial\psi}{\partial x^\lambda}+\left[p_\mu p^\mu-\nabla_\mu p^\mu+\frac{1}{4}\frac{n-2}{n-1}(V-K)\right]\psi=0. \tag{3.6}$$

A special case of (3.6) is

$$V=C, \tag{3.7}$$

which gives us

$$\Box\psi-2p^\lambda\frac{\partial\psi}{\partial x^\lambda}+\left[p_\mu p^\mu-\nabla_\mu p^\mu+\frac{1}{4}\frac{n-2}{n-2}(C-K)\right]\psi=0. \tag{3.8}$$

C is here any constant other than 0, which is not admissible. Equation (3.7) means that for the normalized equation, we have in accordance with (2.10):

$$G+4\frac{n-1}{n-2}a_0=C \tag{3.9}$$

or

$$K+4\frac{n-1}{n-2}(\nabla_\mu p^\mu-p_\mu p^\mu+a_0)=C. \tag{3.10}$$

We may sum up the facts of this paragraph in

Theorem III. *If $g \neq 0$, we may normalize our fundamental equation by means of a multiplicative factor determined as in* (3.2). *The normalized equation determines a unique Riemannian geometry and also a unique vector field superimposed on it.* The scalar field belonging to the equation is no longer independent, but becomes a function of the given tensor and vector fields[18].

§4. **The Normalization of the Dependent Variable.** Besides the point transformations of the coördinates and the multiplication of the fundamental form by an arbitrary function of the coördinates, there is another transformation which leaves invariant the form of the fundamental equation. This is the multiplication of the dependent variable by an arbitrary function of the coördinates. Let this function be S. The fundamental form then becomes

$$F(S\psi) = \Box S\psi - 2p^{\lambda}\frac{\partial S\psi}{\partial x^{\lambda}} + a_0 S\psi. \tag{4.1}$$

We shall not consider this form directly, but the related form

$$S^{-1}F(S\psi) = \Box\psi - \left(2p^{\lambda} - 2S^{-1}g^{\lambda\mu}\frac{\partial S}{\partial x^{\mu}}\right)\frac{\partial\psi}{\partial x^{\lambda}} + \left(a_0 - 2S^{-1}p^{\lambda}\frac{\partial S}{\partial x^{\lambda}} + S^{-1}\Box S\right)\psi \tag{4.2}$$

which leads to the same fundamental equation as (4.1). Under this transformation we have the following relations

$$'g^{\lambda\mu} = g^{\lambda\mu} \tag{4.3}$$

$$'p^{\nu} = p^{\nu} - S^{-1}g^{\nu\mu}\frac{\partial S}{\partial x^{\mu}} \tag{4.4}$$

$$'a_0 = a_0 - 2p^{\lambda}S^{-1}\frac{\partial S}{\partial x^{\lambda}} + S^{-1}\Box S. \tag{4.5}$$

Hence

$$'\nabla_{\mu}\,'p^{\mu} - 'p_{\mu}\,'p^{\mu} + 'a_0 = \nabla_{\mu}p^{\mu} - p^{\mu}p_{\mu} + a_0. \tag{4.6}$$

[18] An alternative theory of normalization has been given by *T. Levi Civita* (loc. cit.). This theory, however, does not seem to be adapted to the needs of quantum theory.

It follows from (4.6) together with (3.10) (which, by the way, completely characterizes our normalization) that this normalization is not in any way influenced by the transformation (4.2). On the other hand, our vector p_μ is modified by the addition of a gradient.

§5. **The Invariant Adjoint Equation.** In the theory of the solution of such partial differential equations as our fundamental equation, a certain related equation known as the *adjoint* is introduced. The equation generally known by this name is, however, not invariantly related to the original equation under the multiplication of the equation by a function of the coördinates and under point transformations of these coördinates. In our invariant theory, we consider the fundamental equation with respect to the general manifold of the x^ν, in which we define a Weyl or Riemannian displacement by the intrinsic properties of the equation. On the other hand, the classical definition of the adjoint presupposes a given Euclidean space in which the x^ν lie. It is only in this space that the classical adjoint has an invariant meaning.

It is, however, a simple matter to define, not only for the equation, but also for the fundamental form, a new adjoint in which the only metric is the intrinsic metric of this form. As in (1.16), we write

$$F(\psi)\equiv\Box\psi-2p^\lambda\frac{\partial\psi}{\partial x^\lambda}+a_0\psi. \tag{5.1}$$

We now consider the new expression

$$G(\psi)\equiv\Box\psi+2p^\lambda\frac{\partial\psi}{\partial x^\lambda}+(2\nabla_\lambda p^\lambda+a_0)\psi. \tag{5.2}$$

We readily deduce the identity for two functions u and z of the x^ν:

$$uF(z)-zG(u)=u\Box z-z\Box u-2p^\lambda\left(u\frac{\partial z}{\partial x^\lambda}+z\frac{\partial u}{\partial x^\lambda}\right)-2(\nabla_\lambda p^\lambda)zu. \tag{5.3}$$

Hence

$$uF(z)-zG(u)=\nabla_\mu v^\mu \tag{5.4}$$

where

$$v_\mu = u\frac{\partial z}{\partial x^\mu} - z\frac{\partial u}{\partial x^\mu} - 2p_\mu uz. \tag{5.5}$$

As $uF(z)-zF(u)$ is the divergence of a vector, we can apply the integral identity

$$\int \nabla_\mu v^\mu dt = \int v^\mu n_\mu d\sigma \tag{5.6}$$

where dt is the invariant volume element of a bounded domain of the n-dimensional manifold, $d\sigma$ the invariant volume element of the $(n-1)$-dimensional boundary, and n^ν the vector perpendicular to this boundary in the sense pertaining to the given metric and directed towards the outside.

Thus it appears that $G(\psi)$ may be considered as the adjoint form of $F(\psi)$. We call it the *invariant adjoint* of $F(\psi)$.[16] In the same way $G(\psi)=0$ is then the invariant adjoint equation of $F(\psi)=0$.

The invariant adjoint of the normalized equation (3.6) is

$$\Box\psi + 2p^\lambda\frac{\partial\psi}{\partial x^\lambda} + \left[p_\mu p^\mu + \nabla_\mu p^\mu + \frac{1}{4}\frac{n-2}{n-1}(V-R)\right]\psi = 0. \tag{5.7}$$

It differs from (3.6) only by the change of the sign of p_μ. Hence the fundamental equation in its invariant form is self-adjoint when and only when $p_\mu = 0$.[17]

Part II. The Physical Theory

§6. **The Fundamental Equation of Wave Mechanics.** It seems that the most elementary phenomenon in nature is what we are accustomed to call the motion of an electron. What we may term the classical notion of the electron envisages it as a small charged particle of matter which in some way or other moves like a very small material sphere under the influence of certain forces. In this paper we shall free ourselves entirely from this conception, and shall follow Schrödinger in defining the

[16] Cotton, l. c., Cf. De Donder-Van Lerberghe, l. c.

[17] If we introduce a connection of the type considered in footnote 10, every fundamental equation is self-adjoint. For such a connection we must replace (5.6) by the integral identity belonging to this connection. See *e.g.*, Schouten, Der Ricci Kalkül. (Berlin, Springer, 1924), Ch. II.

motion of an electron as a wave phenomenon in four independent variables, determined by the equation

$$F(\psi) = g^{\lambda\mu}\frac{\partial^2\psi}{\partial x^\lambda \partial x^\mu} + a_\lambda \frac{\partial\psi}{\partial x^\lambda} + a_0\psi = 0 \tag{6.1}$$

where

$$g = |g^{\lambda\mu}| \neq 0 \tag{6.2}$$

and where the form

$$g_{\lambda\mu}dx^\lambda dx^\mu \text{ has signature } +++-(\text{or}---+) \tag{6.3}$$

Then the equation $F(\psi)=0$ is of the *normal hyperbolic type* (Hadamard).

Under conditions (6.2) and (6.3) the solutions of the fundamental equation are determined by their initial values and the initial values of their derivatives on a given three-dimensional hypersurface which does not contain differential elements determined by the characteristics

$$g_{\lambda\mu}dx^\lambda dx^\mu = 0. \tag{6.4}$$

This means that if we interpret a particular hypersurface of this sort as all space at a particular instant, and take our remaining variable to be the time, the value of ψ, a solution of the fundamental equation, is determined by its values at all points of the world at a particular time and by their rate of change. This is the law of causality which underlies the classical Newtonian physics. No other type of homogeneous linear differential equation than (6.1), with the auxiliary conditions (6.2) and (6.3), is compatible with a causality of this type.

As we have shown in the preceding paragraphs, (6.1) and (6.2) define a Weyl geometry in which $g_{\lambda\mu}$ determines the metrical connection and p_λ the electromagnetic potential. That is, (6.1) and (6.2) yield us a certain uniquely defined quadratic tensor and a certain uniquely defined vector. Einstein has given us a physical interpretation for a tensor of this sort as the gravitational-metrical field of the universe, and Minkowski has shown us how to interpret a vector as an electromagnetic potential. Inasmuch as our tensor and our vector, in accordance with (2.3) and (2.8), are related by the laws of a Weyl geometry, all the arguments which Weyl

adduces to render his world-geometry plausible may be used to support our interpretation of these two quantities.

Furthermore, whereas the Weyl connection is only one of a large number of possible linear connections, so that its preferred position is a little hard to understand, our theory accounts for it as an inevitable consequence of the law of causality of the universe, combined with the assumption that the fundamental facts of nature are of a linear character.

At a later stage of his theory, Weyl returns to the Einstein theory with a fixed $g_{\lambda\mu}$ and a fixed p_λ. To this end he assumes a "gauge equation." This idea may be introduced into wave mechanics in the following way.

Replace, in our theory, this gauge equation by the normalization condition (3.7). We choose this condition as the simplest assumption we can make to determine V. We are manifestly unable to put $V=0$. A more general V would lead to the same physics, but a more complicated mathematics. This is the first point at which the constant C makes its appearance, and here it is absolutely essential. Let us choose the function V so that the normalization takes place in accordance with the formulas (3.8) and (3.10). Let us further assume that in the normalized equation,

$$K=0. \tag{6.5}$$

This is certainly true whenever we may assume, $K_{\lambda\mu}$ being the contracted Riemann-Christoffel tensor belonging to $g_{\lambda\mu}$:

$$K_{\lambda\mu}=0, \tag{6.6}$$

Einstein's law for empty space. We then have, after (3.10):

$$\nabla_\mu p^\mu - p_\mu p^\mu + a_0 = C_1 \,[1] \tag{6.7}$$

where C_1 is a constant given *a priori*. We shall call this constant the *quantization constant* and (6.5) the *quantum condition*. Then the normalized fundamental equation becomes

$$\Box\psi - 2p^\lambda \frac{\partial\psi}{\partial x^\lambda} + [p_\mu p^\mu - \nabla_\mu p^\mu + C_1]\psi = 0, \tag{6.8}$$

[1] For $a_0=0$, we have the normalization equation of the Weyl geometry, if we take K equal to a constant in (6.5). This last assumption, however, does not modify our theory. See A. S. Eddington, l.c. sub[4], p. 209.

and determines a fixed metrical field $g_{\lambda\mu}$, a fixed electromagnetic field p_μ, and a set of solutions ψ determined by these fields and the fixed constant C_1.

In order to bring our work into harmony with existing theories, we assign to C_1 the value

$$C_1 = -\frac{4\pi^2 m^2 c^2}{h^2} \tag{6.9}$$

Here m is the rest-mass of the electron, c the velocity of light, and h the Planck constant. These only occur in the combination $\frac{mc}{h}$, so that at this stage of the theory, this combination alone has a physical meaning. Equation (6.8) has been obtained independently by Fock and De Donder (footnote 3).

§7. **The Quantum Theory of the Single Electron.** Let us now introduce the potential vector ϕ_μ, defined by

$$\phi_\mu = -\frac{ihc}{2\pi e} p_\mu, \tag{7.1}$$

where e is the charge of the electron. This equation merely involves a change of units. Let us furthermore so calibrate our wave-function ψ that

$$\nabla_\mu p^\mu = 0. \tag{7.2}$$

According to §4 this is always possible. Equation (7.2) is known as *Maxwell's supplementary condition*, and fixes the initial values of the potential. The equation of the single electron then assumes the form

$$\Box\psi - \frac{4\pi ie}{hc}\phi^\lambda \frac{\partial\psi}{\partial x^\lambda} + \frac{4\pi^2 e^2}{h^2 c^2}\left(-\phi_\lambda\phi^\lambda - \frac{m^2c^4}{e^2}\right)\psi = 0. \tag{7.3}$$

Under special relativity, (7.3) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{4\pi ie}{hc}\left(\phi_1\frac{\partial\psi}{\partial x} + \phi_2\frac{\partial\psi}{\partial y} + \phi_3\frac{\partial\psi}{\partial z} - \frac{i\phi_4}{c}\frac{\partial\psi}{\partial t}\right)$$
$$+ \frac{4\pi^2 e^2}{h^2 c^2}\left(\phi_4{}^2 - \phi_1{}^2 - \phi_2{}^2 - \phi_3{}^2 - \frac{m^2c^4}{e^2}\right)\psi = 0. \tag{7.4}$$

As there is no longer any difference between covariant and contravariant vectors, we have lowered all indices. Here we have introduced as a new constant c, the velocity of light.

If we put

$$\phi_1 = A_1, \quad \phi_2 = A_2, \quad \phi_3 = A_3, \quad \phi_4 = iV, \tag{7.5}$$

A_1, A_2, A_3 being the components of the magnetic potential, V the electric potential, we see that (7.4) is identical with Schrödinger's relativistic-magnetic generalization of his wave equation.[19] We are hence in a position to annex to our theory all the successes of the Schrödinger theory.

It is to be noted that the first order terms of our generalized Schrödinger equation (and these alone) are pure imaginaries. That is, our equation is the conjugate of its adjoint, or, we may say, is self-adjoint in the Hermitian sense. This fact explains, why it is possible to derive the Schrödinger equation from a minimization problem.

If we put

$$\phi_\lambda = 0, \tag{7.6}$$

we get the equation of the free electron,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{4\pi^2 m^2 c^2}{h^2} = 0. \tag{7.7}$$

Up to now, we have imposed the sole condition on the gravitational field that $K = 0$, and no condition (other than that of realness) on the electromagnetic field. If we put

$$F_{\lambda\mu} = \frac{\partial\phi_\mu}{\partial x^\lambda} - \frac{\partial\phi_\lambda}{\partial x^\mu}, \tag{7.8}$$

we further have without any assumptions

$$\frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0. \tag{7.9}$$

$F_{\lambda\mu}$ is the electromagnetic bivector, and (7.9) constitute the first set of Maxwell equations.

[19] *E. Schrödinger.* Quantisierung als Eigenwertproblem (Vierte Mitteilung). Annalen der Physik (4) Vol. 81. (1926), p. 133, formula (36).

We may follow Einstein in assuming as the law of gravitation outside certain singular points, lines, and regions

$$K_{\lambda\mu}=0. \tag{7.10}$$

This equation is in other words the definition of "empty space." In empty space we assume the second set of Maxwell's equations:

$$\nabla_\mu F^{\lambda\mu}=0. \tag{7.11}$$

Outside of empty space, $K_{\lambda\mu}$ defines the impulse-energy tensor of matter and $\nabla_\mu F^{\lambda\mu}$ defines the current vector.

The simplest non-vanishing set of ϕ_λ which satisfies (7.11) is the Coulomb electric field

$$A_1=0, \quad A_2=0, \quad A_3=0, \quad V=\frac{e}{r}, \quad r^2=x^2+y^2+z^2. \tag{7.12}$$

This gives the correct values for the main energy levels of the hydrogen atom, as Schrödinger himself has shown. It also gives a theory for the relativistic fine structure of the hydrogen lines, but not the correct one, as it leads to azimuthal half quantum numbers. As we have said in the introduction, Epstein has recently developed a theory leading to correct fine-structure levels. His ϕ_λ are

$$\phi_1=0, \quad \phi_2=\frac{ex}{r^3}, \quad \phi_3=\frac{-ey}{r^3}, \quad \phi_4=\frac{e}{r} \tag{7.13}$$

which, like (7.12), satisfy the Maxwell equations except at the origin.[20]

§8. **The Five-Dimensional Theory.** Kaluza[21] has sought to unify the gravitational and electromagnetic fields of particle relativity by combining the second order tensor $g_{\lambda\mu}$ of 16 com-

[20] *E. Schrödinger.* Quantisierung als Eigenwertproblem (Dritte Mitteilung). Annalen der Physik (4) 80 (1926), pp. 437–490, part p. 439 = book cit. sub (7), p. 87.

P. Epstein, l. c. sub (7). The equation with (7.13) was already given by *V. Fock.* Zur Schrödingerschen Wellenmechanik, Zs. für Physik 38 (1926) pp. 242–250.

[21] L.c. sub (8). Comp. also *A. Einstein.* Zu Kaluza's Theorie des Zusammenhanges von Gravitation und Elektrizität. Erste Mitteilung. Sitzungsber. preuss. Akad. Berlin. Physik. Math. Classe 1927, pp. 23–30; and *L. de Broglie.* L'Univers à cinq dimensions et la mécanique ondulatoire. Journal de Physique (6) sec. (1927), pp. 65–73. *L. Rosenfeld.* L'univers à cinq dimensions et la mécanique ondulatoire. Bulletin Classe des Sciences Acad. Roy. Belgique (5) 13 (1927) no. 61.

ponents and the vector ϕ_λ of 4 components with a constant into a symmetric tensor $\gamma_{\lambda\mu}$ of 25 components. This involves the introduction of a new parametric dimension x^5. The equations of definition of the $\gamma_{\lambda\mu}$ are

$$\begin{aligned} \gamma_{\lambda\mu} &= g_{\lambda\mu} + \phi_\lambda \phi_\mu, \quad \lambda, \mu = 1, \cdots, 4 \\ \gamma_{\lambda 5} &= \phi_\lambda \\ \gamma_{55} &= 1, \end{aligned} \tag{8.1}$$

which possess the solutions:

$$\begin{aligned} g_{\lambda\mu} &= \gamma_{\lambda\mu} - \gamma_{5\lambda}\gamma_{5\mu} \\ \phi_\lambda &= \gamma_{\lambda 5} \qquad \lambda, \mu = 1, 2, 3, 4 \\ g &= \gamma = \text{Determinant of the } \gamma_{\lambda\mu}. \end{aligned} \tag{8.2}$$

The Einstein and the second set of Maxwell equations can now be combined into the single minimization principle ($\bar{K}$ being the curvature scalar of the $\gamma_{\lambda\mu}$)

$$\delta \int \bar{K} \sqrt{-\gamma}\, dx^1 dx^2 dx^3 dx^4 dx^5 = 0. \tag{8.3}$$

It should be noted that the tensor $\gamma_{\lambda\mu}$ is not general, but is subject to the conditions that all the components should be independent of x^5, and that γ_{55} should be 1. Thus the symmetry of the combined Einstein-Maxwell principle is largely specious.

Fock and Klein[22] have adapted the Kaluza theory to the needs of wave mechanics. They form the contravariant symmetric tensor $\gamma^{\mu\nu}$:

$$\begin{aligned} \gamma^{\lambda\mu} &= g^{\lambda\mu} \\ \gamma^{\lambda 5} &= -\phi^5 \qquad \lambda, \mu = 1, 2, 3, 4 \\ \gamma^{55} &= 1 + \phi_\lambda \phi^\lambda \end{aligned} \tag{8.4}$$

and define the five-dimensional Laplacian

$$\underset{5}{\Box} u = \frac{1}{\sqrt{-\gamma}} \frac{\partial}{\partial x^\rho} \sqrt{-\gamma}\ \gamma^{\rho\sigma} \frac{\partial u}{\partial x^\sigma}, \qquad \rho, \sigma = 1, 2, 3, 4, 5. \tag{8.5}$$

They then write the wave equation in the form

$$\underset{5}{\Box} u = 0 \tag{8.6}$$

which is homogeneous in order. The transformation

$$u = \psi e^{ix^5}; \quad \psi = \psi(x^1, x^2, x^3, x^4); \tag{8.7}$$

[22] Loc. cit. sub (3) and (8).

reduces this equation to an equation in four variables of the type (2.1).

The transformations of x^1, x^2, x^3, x^4, x^5 which leave the form of our equation unchanged, and preserve a variable filling the role of x^5 are

$$'x^\nu = f_1{}^\nu(x^1x^2x^3x^4), \qquad \nu = 1, 2, 3, 4$$
$$'x^5 = x^5 + f_2(x^1x^2x^3x^4) \tag{8.8}$$

The effect of this transformation on (2.1) is the same as that of a point transformation (1.2) in four-space, accompanied by the transformation

$$'F(\psi) = S^{-1}F(S\psi) \tag{8.9}$$

of §4. On the other hand, the transformations of §2 have no analogue in five dimensions, inasmuch as the multiplication of an expression of the form $\underset{5}{\square} u$ by a function of the form $R(x^1x^2x^3x^4)$ does not leave its form unchanged.

In other words, up to the point where we normalize our equation, there is no pronounced preference to be made between the four-and the five-dimensional theories, although the five-dimensional theory is not so symmetrical as it superficially appears. Our normalization, however, which is the main idea of our theory, and which forms the connecting link between the Einstein field equations and the theory of quanta, and is analogous to Weyl's natural gauge equation, has not been introduced into the five-dimensional theory, and cannot be introduced into this theory without further damaging its symmetry.

It has been shown that in the five-dimensional theory, the motion of an electron in an electromagnetic field may be represented as a projection on the manifold of x^1, x^2, x^3, x^4 of a geodesic line of the five dimensional manifold.[23] This seems to be an advantage, as a simple interpretation of this sort without the aid of a fifth dimension does not seem to be very satisfactory.[24] We may remark in this connection that the whole conception of the motion of a charged particle in a field of force has entirely changed its meaning in wave mechanics, and that the Lorentz equation

[23] *Einstein, Fock, De Broglie*, l. c. sub (8) and (20).

[24] Comp. *L. P. Eisenhart.* The geometry of paths and general relativity. Annals of Math. (2) 24 (1923), pp. 367–392.

of the charged electron has been superseded by our fundamental equation. It can indeed be shown that this equation leads to the Lorentz equation in a limiting case, and this is all that can be expected in the new theory. For if we use the methods suggested by Schrödinger[25] to pass from wave mechanics to particle mechanics and particle electrodynamics, we have from our fundamental equation (6.8) with (7.2)

$$g^{\lambda\mu}\left(\frac{\partial W}{\partial x^{\lambda}}-p_{\lambda}\right)\left(\frac{\partial W}{\partial x^{\mu}}-p_{\mu}\right)+C_1=0 \tag{8.10}$$

and this is the Hamilton equation for the Lorentz electron in general relativity.

§9. **Notes on the Problem of Several Bodies.** In the classical, point-mechanics form of the problem of several bodies, each body is conceived as tracing a locus in a four-dimensional space-time. In the wave mechanics, a body is a phenomenon pervading the whole of space-time. In order to preserve the identity of different bodies, it is apparently necessary to attribute to each a set of space dimensions of its own.[26] If we then wish to hold any form of a theory of relativity, it is hard to see how we can avoid attributing to each body a time of its own as well. That is, the world of the problem of two bodies is an eight-dimensional world.

It is thus natural to seek for the two body analogue of the Schrödinger theory in the theory of the invariants of a second order partial differential equations in eight variables. The development of this theory, however, leads us to suspect that the theory of a single equation may not be adequate, but that an auxiliary equation may be necessary. We are reserving this investigation for a later paper. The whole theory is intimately connected with the theory of the Hamiltonian equation of the problem of two bodies in restricted point relativity. This theory merits a more thoroughgoing investigation than any which it has yet received.

[25] *E. Schrödinger*, l. c. sub (6), p. 133 = book p. 163.

[26] *G. D. Birkhoff*. (A theory of matter and electricity. Proc. Nat. Acad. of Sc. 13 (1927), pp. 160–165) has considered a physics in which the proton and the electron interpenetrate. This view involves problems such as those discussed in this section. Birkhoff's quantum theory, though widely different from ours in detail, also involves a separation of the forces of the universe into three types: the gravitational forces, the electromagnetic forces, and the intra-atomic forces. These correspond in a general way to our $g_{\lambda\mu}$, p_{λ}, and a_0.

One matter of a considerable amount of importance is that of forming some sort of a well-defined four-dimensional space-time from the multi-dimensional world of the problem of several bodies. Perhaps the solution of this problem is to be found in the notion of the exploring particle. If we introduce into a physical system a fictitious particle, affected by the system, but with a mass and charge too slight to influence the system, we may take the intrinsic space and time of this particle as the four-dimensional space-time of the system. Whether this is the true explanation of the four-dimensional character of the world can only be established by a careful examination of its consequences.

A GENERAL OPERATIONAL ANALYSIS

By W. O. PENNELL[1]

§1. **Introduction.** This paper outlines the elements of a general operational analysis. Operational methods have been developed and used by many mathematicians. In recent years, especially, Heaviside has used an operational calculus in connection with the solution of electrical problems.

Broadly speaking, however, mathematicians have not taken readily to operational calculus. One reason has undoubtedly been the feeling that the methods and proof lacked the rigor characteristic of other branches of mathematics. Another reason probably is the fact that a knowledge of operational methods and their power is not very general or wide spread.[2]

§2. **Notation.** The letter p will be the general symbol used for an operator. Different operators in the same problem will be denoted by subscripts, for example p_1, p_2, etc. The operator is always placed to the left of the operand. $p(F)$ represents p operating on the expression F. Without any written operand, p may either represent the symbol of operation alone, or p operating on unity. The context will usually remove any ambiguity of meaning.

The expression $\phi(x)p\psi(x)$ represents the algebraic product of $\phi(x)$ into the result of p operating on $\psi(x)$. Similarly, $p_1p_2\psi(x)$ represents the result of applying p_1 to the result of the operation p_2 on $\psi(x)$.

[1] Chief Engineer, Southwestern Bell Telephone Company, St. Louis, Missouri. The present paper was given as a lecture at the Massachusetts Institute of Technology, April 25, 1927.

[2] For a bibliography of the work on operational methods see Eugene Stephens, Washington University Studies, Vol. XII, Scientific Series, No. 2, 1925, pp. 137–152; S. Pincherle, Équations et opérations fonctionnelles, Encyc. des sci. math., tome II, vol. 5, fasc. 1, 1912. Among more recent papers, cf. J. R. Carson, Bell Telephone Technical Magazine, Oct. 1925, Jan. and April 1926; E. J. Berg, Jour. Franklin Institute, Nov. 1924; J. J. Smith, *ibid.*, Dec. 1925; N. Wiener, Math. Ann. 95 (1926), pp. 557–584; Max Born and Norbert Wiener, this Journal, Feb. 1926; Davis, Am. Jour. of Math., Jan. 1927; P. Lévy, Bull. des sc. math. (2) 50 (1926), pp. 174–192; G. Doetsch, Math. Ann. 89 (1923) pp. 192–207; *idem*, Math. Zeitschr. 22 (1925), pp. 289–306; T. v. Stac Acta lit. ac sci. reg. un. hung. Fran.-Jos., Vol. III, pp. 107–120.

§3. **Definition of Operator.** An operator p is a symbol which when placed to the left of its operand indicates any transformation of the operand which we may define. The operator is defined by a "definition equation" showing its action on the independent variable as follows:

$$p(ax^b) = \psi(a, x, b).$$

As an illustration, if the operator is one which squares the independent variable together with its coefficient, the definition equation is

$$p(ax^b) = (ax^b)^2.$$

If $\psi(x)$ is a function of x which can be expanded in a convergent power series in x, as for example

$$\psi(x) = a_0 + a_1x^{b_1} + a_2x^{b_2} + a_3x^{b_3} + \cdots,$$

we define

$$p\psi(x) = pa_0 + pa_1x^{b_1} + pa_2x^{b_2} + \cdots.$$

Note that if p is defined as a certain operation on ax^b, then $p\psi(x)$ does not in general denote the same operation on $\psi(x)$ as a whole. For example, if $p(ax^b) = (ax^b)^2$, then $p\psi(x)$ is not $[\psi(x)]^2$, but rather

$$a^2{}_0 + (a_1x^{b_1})^2 + (a_2x^{b_2})^2 + \cdots.$$

Since the operator when attached to its operand is an algebraic quantity, the associative and commutative laws of addition apply. For example,

$$p_1\phi(x) + p_2\psi(x) + p_3\Delta(x) = [p_3\Delta(x) + p_1\phi(x)] + p_2\psi(x),$$

and so forth. The symbols of operation are distributive with respect to the operand. That is,

$$[p_1 + p_2 + \cdots]\phi(x) = p_1\phi(x) + p_2\phi(x) + \cdots.$$

This law is true by *definition*. That is, the left side of the above equation is defined as meaning the expression on the right side. In general, however,

$$p[\phi(x) + \psi(x)] \neq p\phi(x) + p\psi(x).$$

Cf. §9.

The symbol of operation is in general not commutative with its operand. That is,

$$p\phi(x) \neq \phi(x)p.$$

For $p\phi(x)$ means the transformation resulting from p operating on $\phi(x)$, while $\phi(x)p$ represents $\phi(x)$ multiplied with the result of p operating on unity, and in general these two operations are not identical.

The symbols of operation in general are not commutative with one another in multiplication. That is,

$$p_1p_2\phi(x) \neq p_2p_1\phi(x).$$

$p_1p_2\phi(x)$ denotes the result of applying p_1 to the result of applying p_2 to $\phi(x)$, while $p_2p_1\phi(x)$ denotes the result of applying p_2 to the result of applying p_1 to $\phi(x)$, and in general the results of the two different sequences of operations are not the same.

§4. **Reciprocal Operators.** $1/p$ or p^{-1} is defined as the inverse operation to p. That is

$$p\cdot(1/p)\phi(x)=\phi(x),$$

or

$$(1/p)\cdot p\phi(x)=\phi(x).$$

For example, let $px^a=ax^{a+1}$. Then

$$(1/p)x^a=x^{a-1}/(a-1).$$

§5. **The Iterated Operator.** If an operator p operates n successive times on its operand, the combined operation is represented by p^n. Thus $ppp\phi(x)=p^3\phi(x)$. Again, if $px^a=ax^{a+1}$, then $p^2x^a=a(a+1)x^{a+2}$.

§6. **Theorem I.** *Any combination of operators can always be represented by a single operator.* For any combination of operators acting on x^a brings about a certain change. This change may be represented by a single operator, since by definition an operator may define any change which we please. For example, let

$$p_1x^a=ax^{a-1};$$
$$p_2x^a=x^{2a}.$$

Then if we put

$$p_3x^a=(p_1+p_2)x^a=p_1x^a+p_2x^a,$$

we have

$$p_3x^a=ax^{a-1}+x^{2a}.$$

§7. **Algebraic Multiplication is an Operational Process.** Since we have defined an operational process as any algebraic transformation acting upon each term of which a function is comprised, it follows that algebraic multiplication is an operational process. Thus if we define p by

$$px^a = x^{a+1},$$

we have

$$p^n x^a = x^n x^a.$$

Quadratic and other algebraic equations may be solved by operational division.

§8. **An Operator Need Not Have the Same Definition Equation Throughout its Range.** For example, we may define an operator by

$$px^a = \psi(a, x)$$

for certain values of a, and for other values, by

$$px^a = \phi(a, x).$$

Again, the definition equation may change with the sequence of terms operated on, being one equation for the first n terms, another equation for the second m terms, etc. As will be seen later, operators which square a given function of x, or cube the function, or take the square root of the function, etc., have a different definition equation for each term operated upon.

§9. **Scalar Commutative Operators.** Some operators are commutative with scalars and some are not. With every scalar commutative operator p,

$$pax^b = apx^b.$$

The definition equation of scalar commutative operators need not show the coefficient of x. Scalar commutative operators have important properties, as will be shown later. An example of a scalar commutative operator is the differential operator p, defined by

$$px^a = ax^{a-1}.$$

It clearly follows from the definition of scalar commutative operators that if p is such an operator,

$$p(a_1+a_2+a_3+\cdots)x^b = pa_1x^b + pa_2x^b + pa_3x^b + \cdots .$$

For

$$\begin{aligned} p(a_1+a_2+\cdots)x^b &= (a_1+a_2+\cdots)px^b \\ &= a_1px^b + a_2px^b + \cdots \\ &= pa_1x^b + pa_2x^b + \cdots . \end{aligned}$$

As a corollary to the above, if the operand of a scalar commutative operator with a single definition equation is expressed as a sum of terms which are functions of the dependent variable, such an operator is distributive as regards these terms. That is,

$$p(\psi(x)+\phi(x)+\cdots) = p\psi(x) + p\phi(x) + \cdots .$$

This is not generally true except for scalar commutative operators.

§10. **Theorem II.** *Polynomials containing only one kind of scalar commutative operator, with integral exponents and constant coefficients, may be multiplied like algebraic quantities.*[3] For example,

$$(ap^2+p)\ (bp^3+cp)\psi(x) = (abp^5+acp^3+bp^4+cp^2)\psi(x).$$

This follows immediately from the distributive law, §3, the definition of a repeated operator in §5, and the definition of a scalar commutative operator in §9.

§11. **Theorem III.** *Given a polynomial containing only one kind of scalar commutative operator with integral exponents and constant coefficients, such as*

$$\psi(p) = a_0 + a_1p^{b_1} + a_2p^{b_2} + \cdots$$

then the reciprocal operator is

$$[\psi(p)]^{-1} = \frac{1}{a_0+a_1p^{b_1}+a_2p^{b_2}+\cdots},$$

where the fraction is defined by algebraic long division of the numerator by the denominator.

[3] Unless otherwise specified, when scalar commutative operators are mentioned, only those with a single definition equation are included.

This long division will give for a quotient a polynomial in p, which we shall consider as an operator. This operator operated upon — that is, by theorem II, multiped by — $\psi(p)$ will equal unity. Hence $\psi(p)$ and $[\psi(p)]^{-1}$ are by definition reciprocal operators when $[\psi(p)]^{-1}$ is defined by algebraic long division.

§12. **Theorem IV.** *Any fraction whose numerator and denominator are polynomials in a scalar commutative operator with integral exponents and constant coefficients is in itself an operator which is defined by long division of numerator and denominator.* That is, if $\psi(p)$ and $\phi(p)$ are such polynomials, $\psi(p)/\phi(p)$ as defined by long division is an operator. For $1/\phi(p)=[\phi(p)]^{-1}$ is the reciprocal operator to $\phi(p)$, by theorem III, and remains an operator when multiplied by $\psi(p)$, by theorem II.

§13. **The Generalized Power of an Operator.** Let us define

$$pbx^a=\psi(b, x, a).$$

Let

$$\phi(b, x, a, n)=\Sigma_r \phi_r(b, a, n)x^r$$

be defined for all real values of n, so that for *integral* values of n

$$p^n bx^a=\phi(b, x, a, n).$$

Let

$$\phi(b, x, a, n_1+n_2)=\Sigma_{v,r} \phi_r(b, a, n_1)\phi(b, x, v, n_2)$$

when n_1 and n_2 are any real numbers. We shall then take $\phi(b, x, a, n)$ to be the *definition* of $p^n bx^a$ for all real values of n. The laws so far given hold for general exponents as well as for integral exponents.

As an example, let

$$px^a=\phi(a)x^{a+k}/\phi(a+k).$$

Then

$$p^n x^a=\phi(a)x^{a+nk}/\phi(a+nk).$$

Again, let

$$px^a=ax^{a-1}.$$

For integral n,

$$p^n x^a=a(a-1)\cdots(a-n+1)x^{a-n}.$$

We may write this

$$p^n x^a=\Gamma(a+1)x^{a-n}/\Gamma(a-n+1).$$

It may be shown that this last formula holds for non-integral

arguments. This operator is of great importance, and has been used by Heaviside in the solution of many electrical problems.

§14. **Theorem V.** *Polynomials containing only one kind of scalar commutative operator with constant coefficients may be multiplied or divided like algebraic quantities.* Since in §13 fractional exponents were so defined as to follow the same laws as integral exponents, theorem V follows directly from theorems II, III, and IV.

§15. **The General Solution of a Linear Operational Equation.** Let us consider the equation

$$p_1y+\psi(x)p_2y+\phi(x)p_3y+\cdots+\theta(x)y=\Delta(x),$$

where $\Delta(x)$, $\psi(x)$, $\phi(x)$, etc. represent analytic functions of x. We may write this equation in the form

$$[p_1+\psi(x)p_2+\phi(x)p_3+\cdots+\theta(x)]y=\Delta.$$

This yields us the operational equation

$$y=\Delta/[p_1+\psi(x)p_2+\phi(x)p_3+\cdots+\theta(x)],$$

where the division is of course operational. To show by an example how this division is carried out, let us consider the equation

$$p^2y+xy=1,$$

where $$px^a=ax^{a+1}.$$

Here $$(1/p)x^a=x^{a-1}/(a-1)$$

and $$(1/p^2)x^a=x^{a-2}/[(a-1)\ (a-2)].$$

We have $$y=1/(p^2+x).$$

This is carried out according to the scheme

$$\begin{array}{l|l}
p^2+x \;\big|\; 1 & \dfrac{1}{p^2}-\dfrac{x^{-1}}{2p^2}+\dfrac{x^{-2}}{2^2\cdot3p^2}-\cdots \\
\quad 1+\dfrac{x^{-1}}{2} & \dfrac{x^{-2}}{2}-\dfrac{x^{-3}}{2^2\cdot3}+\dfrac{x^{-4}}{2^2\cdot3^2\cdot4}-\cdots \\
\quad -\dfrac{x^{-1}}{2} & \\
\quad -\dfrac{x^{-1}}{2}-\dfrac{x^{-2}}{2^2\cdot3} & \\
\quad \dfrac{x^{-2}}{2^2\cdot3} &
\end{array}$$

There are two lines in the quotient, the upper being the quotient expressed operationally, and the lower being the evaluation of the result of applying this operator to 1. We thus get

$$y=\frac{x^{-2}}{2}-\frac{x^{-3}}{2^2\cdot 3}+\frac{x^{-4}}{2^3\cdot 3^2\cdot 4}-\cdots.$$

This is not the complete solution of the original functional equation, but is rather a particular integral. If we had taken

$$px^a=\frac{dx^a}{dx}$$

leaving our other equations unaltered, our method of division would have yielded a particular integral of the differential equation

$$y''+xy=1.$$

This method is capable of extension to the general linear different equation, whether ordinary or partial.[4]

§16. **Proof of the Validity of the Solution by Operational Division.** As above, let Δ, $\psi(x)$, $\phi(x)$, etc., be analytic. Let

$$p_1(y)+\psi(x)p_2(y)+\phi(x)p_3(y)+\cdots+\theta(x)y=\Delta.$$

If we put $\quad p_2(y)=[\psi(x)p_2+\phi(x)p_3+\cdots+\theta(x)]y,$

we have $\quad p_1y+p_2y=\Delta.$

Operational division gives us

$$\underline{p_1+p_2}\,\Big|\,\Delta \quad \Big|\,\frac{\Delta}{p_1}-\left(\frac{1}{p_1}p_2\right)\frac{\Delta}{p_1}+\left(\frac{1}{p_1}p_2\right)^2\frac{\Delta}{p_1}-\cdots+(-1)^n\left(\frac{1}{p_1}p_2\right)^n\frac{\Delta}{p_1}+\cdots$$

$$\underline{\Delta+p_2\left(\frac{\Delta}{p_1}\right)}$$

$$\underline{-p_2\left(\frac{\Delta}{p_1}\right)}$$

$$\underline{-p_2\left(\frac{\Delta}{p_1}\right)-p_2\frac{1}{p_1}p_2\frac{\Delta}{p_1}}$$

$$p_2\frac{1}{p_1}p_2\frac{\Delta}{p_1}$$

[4] Cf. the author's paper, "A New Method for Determining a Series Solution of Linear Differential Equations with Constant or Variable Coefficients." Am. Math. Monthly, June and July, 1926.

The quotient series may or may not converge. Let us suppose that it converges. Let σ_n be its nth partial sum, and let

$$p_1\sigma_n+p_2\sigma_n=S_n.$$

Then

$$\begin{aligned}S_n=\Delta-p_1\left(\frac{1}{p_1}p_2\right)\frac{\Delta}{p_1}+p_1\left(\frac{1}{p_1}p_2\right)^2\frac{\Delta}{p_1}-\cdots+(-1)^{n+1}p_1\left(\frac{1}{p_1}p_2\right)^{n-1}\frac{\Delta}{p_1}\\ +p_2\frac{\Delta}{p_1}-p_2\left(\frac{1}{p_1}p_2\right)\frac{\Delta}{p_1}+\cdots+(-1)^n p_2\left(\frac{1}{p_1}p_2\right)^{n-2}\frac{\Delta}{p_1}\\ +(-1)^{n+1}p_2\left(\frac{1}{p_1}p_2\right)^{n-1}\frac{\Delta}{p_1}\\ =\Delta+(-1)^{n+1}p_2\left(\frac{1}{p_1}p_2\right)^{n-1}\frac{\Delta}{p_1}.\end{aligned}$$

Since the quotient series converges,

$$\lim_{n\to\infty}\left(\frac{1}{p_1}p_2\right)^{n-1}\frac{\Delta}{p_1}=0.$$

It is thus plausible to assume that in the ordinary case,

$$\lim_{n\to\infty}p_2\left(\frac{1}{p_1}p_2\right)^{n-1}\frac{\Delta}{p_1}=0.$$

It follows at once that the result of the operational division satisfies the operational equation. Thus we see that under very general circumstances, if the formal solution by operational division terminates or converges, it yields a valid solution of the operational equation.

§17. **An Example of an Operational Solution.** Let it be required to find the equation of a curve such that the slope of the tangent at any point is equal to the product of the ordinate and abscissa of a point on the curve whose abscissa is equal to one-half the abscissa of the point at which the tangent is drawn. The operational equation is

$$p_1y-\frac{x}{2}p_2y=0$$

where p_1 and p_2 receive the definitions

$$p_1(x^a) = \frac{d}{dx}(x^a);$$

$$p_2(x^a) = \left(\frac{x}{2}\right)^a.$$

We have

$$y = \frac{0}{p_1 - \frac{x}{2}p_2} = \frac{p_1}{p_1 - \frac{x}{2}p_2},$$

since

$$p_1 = \frac{d1}{dx} = 0.$$

We obtain by the process of long division already indicated

$$y = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^5 \cdot 4} + \frac{x^6}{2^{10} \cdot 4 \cdot 6} + \cdots.$$

§18. **Constants of Operation.** The term *constants of operation* is applied to the arbitrary constants which correspond to the constants of integration in differential equations. If the operational symbol is a differential operator then the constants of operation are the constants of integration. Some operational equations have no constants of operation.

In general we may expect constants of operation when p is such an operator that a value of a can be found which makes $pAx^a = 0$ vanish where A is any constant. If, for example, p is the differential operator, then when $a = 0$, pAx^a vanishes.

The usual method of bringing the constants of operation into the solution is as follows: let

$$p^n y + p^{n-1} y + \cdots + py = \phi(x)$$

be an operational equation. Then

$$y = \phi(x)/p^n + p^{n-1} + \cdots + p). \qquad (a)$$

Suppose that

$$pA_1 x^k = p^2 A_2 x^k = \cdots = p^n A_n x^k = 0.$$

We may then write our equation in the form

$$y = [\phi(x) + pA_1 x^k + \cdots + p^n A_n x^k]/(p^n + p^{n-1} + \cdots + p)$$

as we have merely added to the numerator of our relation (*a*) quantities which are equal to zero. If we now perform the operational division indicated in the last form of our equation, the n constants of operation $A_1, \cdots, A_n$ appear.[8]

Some illustrations will be given:

Example 1.

$$y''+xy'+y=1,$$

which we may write

$$p^2y+xpy+y=1,$$

with solution

$$y=(1+p^2A+pB)/(p^2+xp+1).$$

Example 2.

$$p^2y+py+xy=x^2$$

(where

$$px^a=(a-2)x^{a-1}1)$$

Here

$$px^2=p^2x^2=0.$$

The solution is

$$y=(x^2+pAx^2+p^2Bx^2)/p^2+p+x.$$

Example 3. Let z be a function of x and y. Consider the equation

$$\partial^2z/\partial x^2+x\partial^2z/\partial x\partial y+xz=1$$

and let p_x and p_y represent differentiation with respect to x and y, respectively. The equation then becomes

$$p_x^2z+xp_xp_yz+xz=1.$$

We have

$$z=(1+p_x^2\psi(y)+p_x\phi(y))/p_x^2+xp_xp_y+x.$$

Here $\psi(y)$ and $\phi(y)$ are constants with respect to x.

§19. **Operational Division: Arrangement of Terms in Divisor.** (*a*) When the divisor contains only one kind of operator p, different results will be obtained according to whether the terms of the divisor are arranged in descending or in ascending powers of p. Very often one result will be a convergent series and consequently

[8] Sometimes a different process of obtaining the first term of the power solution by an index equation, and thus obtaining the arbitrary constants for the dividend by working backwards, gives a simpler division operation.

a solution, and the other result will be a divergent series. Sometimes both results are convergent or finite, and both are solutions.

(*b*) When the divisor contains several different kinds of operators p_1, p_2, p_3, etc., different results will be obtained depending upon which operator, p_1, p_2, etc., is the first term of the divisor. The sequence of the other terms is immaterial. A finite result of a convergent series is in general a solution.

§20. **Example.** A unit mass moves at an irregularly accelerated velocity under the influence of a varying force. The acceleration at any time t is equal to the velocity of time t^2. We are to find the distance covered in time t, and the expression of the force as a function of t.

We put $s=\psi(t)$ as the distance covered in time t. p_1 is the operation of differentiating with respect to the time, and the velocity is $p_1 s$. We write p_2 for the operator defined by the equation

$$p_2 t^a = (t^2)^a = t^{2a}.$$

The operational equation then becomes

$$p_1^2 s - p_1 p_2 s = 0$$

which gives

$$s = A p_1^2/(p_1^2 - p_2 p_1) \tag{1}$$

and

$$s = B p_1/(p_1^2 - p_2 p_1). \tag{2}$$

The solution of (1) is

$$\begin{array}{l|ll} p_1^2 - p_2 p_1 & A p_1^2 & A \\ & A p_1^2 - 0 & \end{array}$$

while that of (2) is

$$\begin{array}{l|l} p_1^2 - p_2 p_1 & \dfrac{B p_1}{p_1^2} + \dfrac{B}{p_1^2} + \dfrac{Bt^2}{p_1^2} + \dfrac{Bt^4}{3p_1^2} + \cdots \\ \quad B p_1 & \\ \quad B p_1 - B & Bt + \dfrac{Bt^3}{2} + \dfrac{Bt^4}{3\cdot 4} + \dfrac{Bt^5}{3\cdot 7\cdot 8} + \cdots \\ \quad\quad B & \\ \quad\quad B - Bt^2 & \\ \quad\quad\quad Bt^2 & \\ \quad\quad\quad Bt^2 - \dfrac{Bt^4}{3} & \\ \quad\quad\quad\quad \dfrac{Bt^4}{3} & \end{array}$$

Thus

$$S=A+B\left[t+\frac{t^2}{2}+\frac{t^4}{3\cdot 4}+\frac{t^5}{3\cdot 7\cdot 8}+\cdots\right];$$

$$F=\frac{d^2S}{dt^2}=B\left[1+t^2+\frac{t^6}{3}+\cdots\right].$$

§21. **Example.** We are to find a power expansion in terms of x whose first differential coefficient is a series whose first, third, fifth etc. terms are the squares of the corresponding terms in the power expansion, and whose second, fourth, sixth etc. terms are equal to the corresponding terms of the power expansion. We shall write $y=\psi(x)$ for the power expansion, p_1 for the differential operator, and p_2 for the operator which has the effect

$$p_2(ax^b)=a^2x^{2b}$$

when acting on the odd terms, and

$$p_2(ax^b)+ax^b$$

when acting on the even terms. We have

$$y=Ap_1/(p_1-p_2).$$

The long division is

$$\begin{array}{l|l}
p_1-p_2 & Ap_1 \\
\hline
 & \underline{Ap_1-A^2} \\
 & \quad A^2 \\
 & \quad \underline{A^2-A^2x} \\
 & \qquad A^2x \\
 & \qquad\quad \underline{A^2x-\frac{A^4x^4}{2^2}} \\
 & \qquad\qquad \frac{A^4x^4}{2^2} \\
 & \qquad\qquad \frac{A^4x^4}{2^2}-\frac{A^4x^5}{2^2\cdot 5}
\end{array}$$

$$\begin{array}{l}
A+\frac{A^2}{p_1}+\frac{A^2x}{p_1}+\frac{A^4x^4}{2^2\cdot p_1}+\frac{A^4x^5}{2^2\cdot 5\cdot p_1}+\cdots \\
\hline
A+A^2x+\frac{A^2x^2}{2}+\frac{A^4x^5}{2^2\cdot 5}+\frac{A^4x^6}{2^2\cdot 5\cdot 6}+\cdots
\end{array}$$

and we get

$$y=A+A^2x+\frac{A^2x^2}{2}+\frac{A^4x^5}{2^2\cdot 5}+\frac{A^4x^6}{2^2\cdot 5\cdot 6}+\cdots$$

§22. **The Operator which Squares a Function.** It may be of value to have the operator which squares the entire function. That is, if

$$y=\psi(x),$$

we are to have

$$py=y^2=[\psi(x)]^2.$$

When this operator acts on the first term a of the quantity to be squared, it must yield a^2. When it operates on the second term b, it must yield b^2+2ab. Similarly, when operating on the third term c, it must yield $c^2+2ac+2bc$, and so on. Let us call this operator p_2 in the following example, and let the differential operator be p_1. Then the differential equation

$$y''+y^2=x$$

may be written

$$p_1^2y+p_2y=x.$$

The division is

$$\begin{array}{l|l}
p_1^2+p_2 \;\big|\; x & \dfrac{x}{p_1^2}-\dfrac{x^6}{(3!)^2p_1^2}+\dfrac{2x^{11}}{(3!)^3\cdot7\cdot8p_1^2}+\cdots \\
\quad x+\dfrac{x^6}{(3!)^2} & \\
\hline
\quad -\dfrac{x^6}{(3!)^2} & \dfrac{x^3}{3!}-\dfrac{x^8}{(3!)^2\cdot7\cdot8}+\dfrac{2x^{13}}{(3!)^3\cdot7\cdot8\cdot12\cdot13}+\cdots \\
\quad -\dfrac{x^6}{(3!)^2}-\dfrac{2x^{11}}{(3!)^2\cdot7\cdot8}+\dfrac{x^{16}}{(3!)^4\cdot7^2\cdot8^2} & \\
\hline
\quad \dfrac{2x^{11}}{(3!)^2\cdot7\cdot8}-\dfrac{x^{16}}{(3!)^4\cdot7^2\cdot8^2} &
\end{array}$$

and the particular integral is

$$y=\frac{x^3}{3!}-\frac{x^8}{(3!)^2\cdot7\cdot8}+\frac{2x^{13}}{(3!)^3\cdot7\cdot8\cdot12\cdot13}+\cdots.$$

Operations which cube a function or take its square root may be used, the definition equations being obtained in a manner similar to that shown above. Theoretically, we may introduce operators performing many other operations on the function as a whole. In many cases, however, such operations become cumbersome and involved.

§23. **Examples of Special Operators.** Let us consider

$$y''+(y')^2+xy=x.$$

We may write this

$$p_1^2y+p_2p_1y+xy=x$$

where p_1 is the differential operator, and p_2 is the operator which squares a function.

$$y=x/(p_1^2+p_2p_1+x)$$

is a particular solution, yielding

$$y=\frac{x^3}{3!}-\frac{x^6}{72}+\frac{7x^9}{(72)^2}-\frac{53x^{12}}{6\cdot11\cdot(72)^2}+\cdots\cdot$$

The complete solution is

$$y=(x+Ap_1^2+Bp_1)/(p_1^2+p_2p_1+x).$$

yielding

$$y=A+Bx+\frac{B^2x^2}{2!}+\frac{(1-A+2B_3)}{3!}x_3-\frac{2B-AB+3B^4}{3\cdot4}x^4+\cdots\cdot$$

As may be readily seen, if we put the constants of integration equal to 0, we obtain the particular solution already given.

Again, let us consider

$$y''+xy^{1/2}=1$$

which becomes

$$p_1^2y+xp_2y=1$$

when p_1 is the differential operator and p_2 is the operator which takes the square root of a function. We have

$$y=(1+Ap_1^2+Bp_1)/(p_1^2+xp_2),$$

which yields us as a result of long division the particular solution

$$y=\frac{x^2}{2!}-\frac{x^4}{3\cdot2^{5/2}}+\frac{x^6}{6!}+\frac{x^8}{3^2\cdot5\cdot7\cdot2^5}+\cdots\cdot$$

MULTIPLE INVARIANTS AND GENERALIZED RANK OF A P-WAY MATRIX OR TENSOR

By Frank L. Hitchcock

TABLE OF CONTENTS

PAGE

1. Multiply invariant properties of a matrix. 40
2. Multiply covariant matrices 41
3. Vacant and filled indices 43
4. The significance of multiple covariants 43
5. The relation of multiple covariants to covariants of algebraic forms 44
6. Polyadic or symbolic representation of multiple covariants 45
7. Properties of the multiple covariants $\mathbf{F}_k((\mathbf{A}^n))$ 46
8. Extension of the concept of minor 49
9. Multiple concomitants in general. 50
10. Properties of the multiple concomitants $\mathbf{F}_k((\mathbf{A}^q))$ 50
11. Further extension of the concept of minor 51
12. Folding on arbitrarily chosen indices 52
13. Extension of the concept of simple rank. 53
14. Explicit formulation of the matrix $[\mathbf{A}^q]^{A\cdots G}$ 54
15. Extension of the concept of multiplex rank 55
16. Multiplex space complement 56
17. Multiplex folding 60
18. Rank of various grades 64
19. Further examples of multiple concomitants 65
20. Multiple concomitants and tensor calculus 66
21. Illustrative examples 70

MULTIPLE INVARIANTS AND GENERALIZED RANK OF A P-WAY MATRIX OR TENSOR

By FRANK L. HITCHCOCK

1. Multiply invariant properties of a matrix.

In former papers[1] arose properties of a matrix $A_{i_1 \cdots i_p}$ which persist after any non-singular linear transformation on an arbitrarily chosen index i_m

$$A'_{i_1 \cdots i_{m-1}\, i_{(m)} i_{m+1} \cdots i_p} = A_{i_1 \cdots i_{m-1} \alpha_m i_{m+1} \cdots i_p}\, T_{\alpha_m i_m} \qquad (1)$$

summed on the Greek index from *1* to *n*. First of all are multiple invariants in the narrow sense, polynomials homogeneous in the $A_{i_1 \cdots i_p}$ which, under (1), are merely multiplied by a power of the determinant Δ of **T**. That is, if $F(A_{i_1 \cdots i_p})$ is a polynomial homogeneous in the original components $A_{i_1 \cdots i_p}$, and if $F(A'_{i_1 \cdots i_p})$ is the same function of the new components obtained by the transformation (1), the condition that F shall be a multiple invariant of **A** is

$$F(A'_{i_1 \cdots i_p}) = \Delta^w F(A_{i_1 \cdots i_p}) \qquad (2)$$

for every **T** and for every choice of the index i_m. (See IV, p. 246 and V, p. 166.)

The two following properties have also been established:

(A) The rank, simple or multiplex, of the matrix $A_{i_1 \cdots i_p}$ on any index or on any group of indices is a multiply invariant number, *i.e.*, it is the same for the new matrix $A'_{i_1 \cdots i_p}$ (V, p. 177).

(B) The possibility of expressing the matrix $A_{i_1 \cdots i_p}$ as a sum of products

$$A_{i'_1 \cdots i_p} = \sum_{j=1}^{j=h} B_{1j,\, u_1}\, B_{2j,\, u_2} \cdots B_{\pi j,\, u_\pi} \qquad (3)$$

[1] I. *Determinantal Expression of Multiple Products*, etc., by Lepine Hall Rice (this Journal, Vol. IV, No. 2, p. 130, April 1925).
II. *The Multiple Complement of One or More Polyadics*, Rice and Hitchcock (this Journal, Vol. IV, No. 3, p. 179, May 1925).
III. *A Theory of Ordered Determinants* (Vol. IV, No. 4, p. 205).
IV. *A New Method in the Theory of Quantics* (Ibid., p. 238).
V. *The Expression of a Tensor or a Polyadic as a Sum of Products*, (Vol. VI, No. 3, p. 164, April 1927).
These papers will be referred to by the Roman numerals as listed.

is a multiply invariant property of the matrix, given the value of h. Here the indices $u_1 \cdots u_\pi$ are either simple or multipartite; so that the B's denote either vectors or matrices (polyadics, tensors), of class agreeing with the multipartite character of their respective indices (V, p. 174).

2. Multiply covariant matrices.

Closely related to (A) and (B) are matrices whose components, considered separately, are not multiple invariants, but which, taken as a whole, possess multiply invariant properties. Let there be a matrix $\mathbf{F}$ of class r with components $F_{j_1 \cdots j_r}$ rational, integral and homogeneous of degree q in the $A_{i_1 \cdots i_p}$ or else zero. Let $\overline{\mathbf{F}}$ be a new matrix, also of class r, whose components are the same functions of the $A'_{i_1 \cdots i_p}$ that the $F_{j_1 \cdots j_r}$ were of the untransformed $A_{i_1 \cdots i_p}$. Now on the other hand let a matrix $\mathbf{F}'$ be obtained by transforming $\mathbf{F}$ in the following manner: each of the r indices of $\mathbf{F}$ is to be *associated with* or *belong to* a particular index of $\mathbf{A}$; it may be that all indices of $\mathbf{F}$ belong to the same index of $\mathbf{A}$, or some of the $j_1 \cdots j_r$ belong to one of the $i_1 \cdots i_p$, others to another; when the arbitrary index i_m of $\mathbf{A}$ on which the transformation (1) is to occur has been selected, the same transformation is then carried out simultaneously on those indices of $\mathbf{F}$, if any, which belong to i_m, *but on no others:* the resulting matrix is $\mathbf{F}'$.

Definition I. If

$$\overline{\mathbf{F}} = \Delta^n \mathbf{F}' \tag{4}$$

I shall say that $\mathbf{F}$ is a *multiple covariant* of $\mathbf{A}$.

Equation (4) is the same as

$$\overline{F}_{j_1 \cdots j_r} = \Delta^n F'_{j_1 \cdots} \tag{4a}$$

constituting, for a particular choice of i_m, a system of n^r equations; and since there are p choices of i_m the definition imposes pn^r conditions upon the components of $\mathbf{F}$.

As an illustration let $n=2$, $p=3$, *i.e.*, let $\mathbf{A}$ be a binary triadic

$A_{i_1 i_2 i_3}$, and let $\mathbf{F}$ be a dyadic $F_{j_1 j_2}$ whose components are the cubic determinants

$$\begin{vmatrix} A_{11j_1} & A_{12j_1} \\ A_{21j_1} & A_{22j_1} \end{vmatrix} \quad \begin{vmatrix} A_{11j_2} & A_{12j_2} \\ A_{21j_2} & A_{22j_2} \end{vmatrix} \tag{5}$$

that is

$$F_{j_1 j_2} = A_{11j_1}A_{22j_2} - A_{12j_1}A_{21j_2} - A_{21j_1}A_{12j_2} + A_{22j_1}A_{11j_2} \tag{6}$$

so that

$$\begin{aligned} F_{11} &= 2(A_{111}A_{221} - A_{121}A_{211}), \quad F_{22} = 2(A_{112}A_{222} - A_{122}A_{212}) \\ F_{12} &= F_{21} = A_{111}A_{222} - A_{121}A_{212} - A_{211}A_{122} + A_{221}A_{122}. \end{aligned} \tag{7}$$

(These are the quantities (33) of V, p. 179, *i.e.*, $\mathbf{F}$ is the $\mathbf{F}_1((\mathbf{A}^2))$ of that paper). Both indices of $\mathbf{F}$ belong to i_3 of $\mathbf{A}$.

To verify by direct calculation that $\mathbf{F}$ satisfies (4) take first $i_m = i_1$. Then $\mathbf{F}' = \mathbf{F}$ because no index of $\mathbf{F}$ belongs to i_1 and so $\mathbf{F}$ is by definition untransformed. On the other hand $\overline{\mathbf{F}} = \Delta\mathbf{F}$; for by definition

$$\begin{aligned} \overline{F}_{j_1 j_2} &= \begin{vmatrix} A_{a_1 1 j_1} T_{a_1 1} & A_{a_1 2 j_1} T_{a_1 1} \\ A_{a_1 1 j_1} T_{a_1 2} & A_{a_1 2 j_1} T_{a_1 2} \end{vmatrix} \begin{vmatrix} A_{a_1 1 j_2} T_{a_1 1} & A_{a_1 2 j_2} T_{a_1 1} \\ A_{a_1 1 j_2} T_{a_1 2} & A_{a_1 2 j_2} T_{a_1 2} \end{vmatrix} \\ &= \Delta F_{j_1 j_2}, \end{aligned} \tag{8}$$

i.e., the 3-way determinant (8) is equal to the product of the 2-way determinant Δ into the 3-way determinant (5), by the Cayley-Rice law of multiplication. Thus (4) is satisfied with $\mu = 1$ when $i_m = i_1$.

Next let $i_m = i_2$. The proof is similar with again $\mu = 1$. Finally letting $i_m = i_3$ we have by definition

$$\overline{F}_{j_1 j_2} = \begin{vmatrix} A_{11a_3} T_{a_3 j_1} & A_{12a_3} T_{a_3 j_1} \\ A_{21a_3} T_{a_3 j_1} & A_{22a_3} T_{a_3 j_1} \end{vmatrix} \begin{vmatrix} A_{11a_3} T_{a_3 j_2} & A_{12a_3} T_{a_3 j_2} \\ A_{21a_3} T_{a_3 j_2} & A_{22a_3} T_{a_3 j_2} \end{vmatrix} \tag{9}$$

On the other hand the transformed components $F'_{j_1 j_2}$ are $F_{a_1 a_2} T_{a_1 j_1} T_{a_2 j_2}$ and are identical with (9) because, in a cubic determinant, a factor of every element of a layer is a factor of the determinant. (Note that a_1, a_2, and a_3 are dummies.) Therefore $\overline{\mathbf{F}} = \mathbf{F}'$ and (4) is satisfied with $\mu = 0$ when $i_m = i_3$. It has thus been shown that $\mathbf{F}$ is a multiple covariant of $\mathbf{A}$.

3. **Vacant and filled indices.**

Let an index of **A** which has no index of **F** associated with it be called a *vacant index*, and an index of **A** which has one, two, or k indices of **F** associated with it be called *one-filled*, *two-filled*, or *k-filled*.

When the transformation (1) occurs on a vacant index, **F** is by definition untransformed, *i.e.*, $\mathbf{F}'=\mathbf{F}$ (as in the case $i_m=i_1$ above), so that $\overline{\mathbf{F}}=\Delta^n\mathbf{F}$ if (4) is satisfied. This is the same as saying that *each component* of **F** is an invariant of **A** on the index in question. We may thus state

Theorem I. That a matrix **F** may satisfy the conditions of multiple covariance with respect to the vacant indices of **A** it is necessary and sufficient that each component of **F** shall be an invariant of **A** on each vacant index.

Since $\overline{\mathbf{F}}$ is of the same degree in the components of **T** as in those of **A**, and since the right member of the equation $\overline{\mathbf{F}}=\Delta^n\mathbf{F}$ is of degree μn in those of **T**, it follows that $\overline{\mathbf{F}}$ must be of degree μn in those of **A** and we have

Theorem II. If **F** is a multiple covariant of **A** and if a vacant index exists, **F** is of degree μn in the components of **A** where μ is a positive integer.

Corollary. If several vacant indices exist, μ has the same value for all.

But if an index i_m is k-filled $\mathbf{F}'$ is of degree k in the components of **T**. Hence by comparing both sides of (4) we may write

Theorem III. If **F** is a multiple covariant of **A** and if a k-filled index exists, the components of **F** are of degree $\mu n+k$ in those of **A**, where μ is a positive or a negative integer or zero.

As an obvious consequence follows the

Corollary. If an index is k-filled and another is k'-filled, k and k' differ at most by a multiple of n.

4. **The significance of multiple covariants.**

The utility of multiple covariants is largely a consequence of the following theorem:

Theorem IV. Any multiply invariant property possessed by $\mathbf{F}$ is also possessed by $\overline{\mathbf{F}}$.

Proof. The assumed property is possessed by $\mathbf{F}'$ because the transformation of $\mathbf{F}$ into $\mathbf{F}'$ on the k indices which belong to i_m may be carried out in k successive steps one for each index. Multiplication by Δ^μ may be regarded as a linear transformation on one index. Thus $\mathbf{F}$ can be converted into $\overline{\mathbf{F}}$ by a succession of transformations each of which pertains to a single index. Hence the assumed property is possessed by $\overline{\mathbf{F}}$.

Corollary I. The rank of $\mathbf{F}$, simple or multiplex, on any index or group of indices, is a multiply invariant number belonging to $\mathbf{A}$.

Corollary II. If $\mathbf{F}$ vanishes in all components so does $\overline{\mathbf{F}}$.

5. The relation of multiple covariants to covariants of algebraic forms.

We may suppose the $A_{i_1 \cdots i_p}$ to be coefficients of a multilinear algebraic form

$$A_{i_1 \cdots i_p} x_{i_1}^{(1)} \cdots x_{i_p}^{(p)} \text{ (summed on all indices)} \qquad (10)$$

where the x's are p sets of n variables each. The $F_{j_1 \cdots j_r}$, similarly, may determine another multilinear form

$$F_{j_1 \cdots j_r} y_{j_1}^{(1)} \cdots y_{j_r}^{(r)} \text{ (summed on all indices)} \qquad (11)$$

where the y's are r other sets of n variables each. An arbitrary transformation (1) on i_m is equivalent to an equally arbitrary transformation on *one* set of variables $x_{im}^{(m)}$. If i_m is a vacant index no set of y's is affected but if i_m is k-filled k sets of y's are cogredient. Any set of y's is cogredient with one and only one set of x's. "Whether we think of such forms, or of the matrices, or of some physical or geometrical entity determined by them is, from a purely algebraic point of view, wholly a question of taste."[2]

If, instead of requiring the covariant property to hold for each separate index i_m we had adopted the less stringent condition that *all* variables be simultaneously transformed by the same $\mathbf{T}$, then

[2] Freely translated from the *Invariantentheorie* of Dr. Roland Weitzenbock Noordhoff 1923) s. 253.

F would be a covariant of the given algebraic form (10) in the sense of the classical theory, and would also be a covariant of **A** as a polyadic (IV, p. 249). In other words a *multiple covariant* is a *covariant*, though of special type.

If, less generally, we were not only to suppose all the x's and all the y's to be cogredient but were also to make all these $p+r$ sets the *same* set (in which case the matrices **F** and **A** may be taken as symmetrical) the form (10) reduces to a quantic of order p in one set of variables and (11) to an ordinary covariant of a single quantic.

On the other hand, as a supposition closer to the present investigation, we might make both **F** and **A** symmetrical while still maintaining the $p+r$ sets of variables distinct and not in general cogredient. If then **F** is a *multiple covariant* of **A** it is a *covariant* of **A** regarded as a quantic of order p. It follows that the covariants of a single quantic may be classified according to whether or not they are *multiple* covariants.

In any case, however, the present discussion assumes a definite *order* to exist among the indices or sets of variables, as well as a definite order from *1* to n among the variables of a set. In a sense this assumption, if we abstract from all arithmetical and geometrical connotations, makes the form (10) equivalent to a polyadic; and polyadics may be said to arise by adopting, for economy, a definite order of writing instead of upper indices, just as, in decimal arithmetic, positional significance replaces denominational symbols; then, to emphasize non-commutativity, $\mathbf{e}_1 \cdot\cdot \mathbf{e}_n$ are written for $x_1 \cdot\cdot x_n$ giving

$$\mathbf{A} = \mathrm{A}_{i_1 \cdot\cdot i_p} \mathbf{e}_{i_1} \cdot\cdot \mathbf{e}_{i_p} \quad \text{(summed on all indices)} \qquad (12)$$

and similarly for **F**.

6. Polyadic or symbolic representation of multiple covariants.

By whichever of various possible methods we prefer to establish relationship between matrices, forms, and polyadics,[3] we may

[3] *e.g.*, we might *begin* with (13), defining the vector factors of any one index as non-commutative symbols of which any $n+1$ are linearly related; then choose, for each value of j, a set of n not so related to serve as the basis vectors of that index; and obtain the $\mathrm{A}_{i_1 \cdot\cdot i_p}$ by resolving into components. Or, beginning with $\mathrm{A}_{i_1 \cdot\cdot i_p}$, we might define addition and general multiplication as in V, p. 164 and basis vectors as in V, p. 166. Or again we may regard the polyadic as due merely to omission of indices, general product being a consequence of ordinary multiplication. All these are but aspects of one system.

always for large enough h write as a special case of (3)

$$\mathbf{A}=\mathbf{a}_1\mathbf{a}_2\cdot\cdot\,\mathbf{a}_p+\mathbf{b}_1\mathbf{b}_2\cdot\cdot\,\mathbf{b}_p+\cdot\cdot\quad\text{to h terms} \tag{13}$$

and similarly for $\mathbf{F}$. The m.i.p. product (III, p. 212) of $\mathbf{A}$ into itself, denoted by $[\mathbf{AA}]$ or $[\mathbf{A}^2]$ is then given by

$$[\mathbf{A}^2]=\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2\cdot\cdot\,\mathbf{a}_p\mathbf{b}_p \tag{14}$$

omitting symbols of summation when no confusion results. The right member may by analogy be called the *symbolic* representation of $[\mathbf{A}^2]$, and similarly in all cases.[4] In matrix notation if the left member be called $C_{i_1j_1i_2j_2\cdot\cdot i_pj_p}$ the equivalent of (14) is

$$C_{i_1j_1\cdot\cdot i_pj_p}=A_{i_1\cdot\cdot i_p}A_{j_1\cdot\cdot j_p}. \tag{14a}$$

The utility of the polyadic or symbolic method is exemplified by the ease with which we may establish such theorems as the following:

Theorem V. The polyadics $\mathbf{F}_k((\mathbf{A}^n))$ are multiple covariants of $\mathbf{A}$.

Proof. By definition (III, p. 228) the required polyadics are obtained from $[\mathbf{A}^n]$ by folding all but the last k factors. It will be sufficient to carry out the case $n=4$, $p=5$, $k=3$, the proof being in all cases similar. We have symbolically

$$\mathbf{F}_3((\mathbf{A}^4))=(\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1)(\mathbf{a}_2\mathbf{b}_2\mathbf{c}_2\mathbf{d}_2)\mathbf{a}_3\mathbf{b}_3\mathbf{c}_3\mathbf{d}_3\mathbf{a}_4\mathbf{b}_4\mathbf{c}_4\mathbf{d}_4\mathbf{a}_5\mathbf{b}_5\mathbf{c}_5\mathbf{d}_5 \tag{15}$$

evidently a polyadic of order 12 whose first four indices belong to i_3, the next four to i_4, and the rest to i_5. The indices i_1 and i_2 are vacant. The multiple covariant property is evident by inspection: if $i_m=i_1$ we have $\mu=1$ because $(\mathbf{a}'_1\mathbf{b}'_1\mathbf{c}'_1\mathbf{d}'_1)=\Delta(\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1)$; if $i_m=i_3$, $\mu=0$ since

$$\overline{\mathbf{F}}=\mathbf{F}'=(\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1)(\mathbf{a}_2\mathbf{b}_2\mathbf{c}_2\mathbf{d}_2)\mathbf{a}'_3\mathbf{b}'_3\mathbf{c}'_3\mathbf{d}'_3\mathbf{a}_4\mathbf{b}_4\mathbf{c}_4\mathbf{d}_4\mathbf{a}_5\mathbf{b}_5\mathbf{c}_5\mathbf{d}_5;$$

and similarly in all cases.

7. Properties of the multiple covariants $\mathbf{F}_k((\mathbf{A}^n))$.

The polyadics of Theorem V make up the simplest class of multiple covariants having vacant indices with $\mu=1$. They evi-

[4] *i.e.*, vector factors take the place of symbolic or ideal factors which are no longer necessary, — or we may say that ideal factors are given a new interpretation.

dently constitute a special case of the polyadics $\mathbf{F}_k((\mathbf{P}, \mathbf{Q}, \cdots))$ which were studied in detail in III. In the notation of that paper, if $\mathbf{A}$ be expanded on its last k factors as

$$\mathbf{A} = \mathbf{e}_{a_1} \cdots \mathbf{e}_{a_{p-k}} \mathbf{M}_{a_1 \cdots a_{p-k}} \tag{16}$$

where the $\mathbf{M}$'s are a set of n^{p-k} polyadics of order k, it was shown that $\mathbf{F}_k((\mathbf{A}^n))$ may be written as an ordered determinant of class $p-k+1$ and order n with all layers of the first direction alike. The elements of any layer of the first direction are the $\mathbf{M}_{a_1 \cdots a_{p-k}}$; the development is by m.i.p.

It was also shown that the components of $\mathbf{F}_k((\mathbf{A}^n))$ are themselves determinants of class $p-k+1$ and order n each element of which is one of the components $A_{i_1 \cdots i_p}$ of the original matrix. To express any component of $\mathbf{F}_k((\mathbf{A}^n))$ in the language of matrices, the last k indices of $A_{i_1 \cdots i_p}$ are assigned fixed values (the fixed indices $r_i s_i \cdots z_i$ of III, p. 231); each element $A_{i_1 \cdots i_p}$ carries an upper index thus

$$A^{(i)}_{i_1 \cdots i_p}$$

which is a dummy because the polyadics $\mathbf{P}$, $\mathbf{Q}$, $\cdots$ are all equal to $\mathbf{A}$; this dummy, together with the first $p-k$ indices of $A_{i_1 \cdots i_p}$ give the $p-k+1$ "directions" of the required component of $\mathbf{F}$, and, as they run from *1* to n, yield the required n^{p-k+1} elements of this component. In matrix or tensor language we may therefore write

$$\mathbf{F}_{j_1 \cdots j_{kn}} =$$

$$\Sigma \pm A^{(1)}_{i_1 \cdots i_{p-k} j_1 \cdots j_k} A^{(2)}_{i_1 \cdots i_{p-k} j_{k+1} \cdots j_{2k}} \cdots A^{(n)}_{i_1 \cdots i_{p-k} j_{nk-k+1} \cdots j_{nk}} \tag{17}$$

where the upper numbers $(1) \cdots (n)$ denote n choices of the values of the indices $i_1 \cdots i_{p-k}$, which are signant.

The matrices, polyadics, tensors, or multilinear forms $\mathbf{F}_k((\mathbf{A}^n))$ are fully determined by (17), which might have been taken as the definition of these quantities, irrespective of the folding or symbolic process by which they were originally obtained. The illustration (6) of Art. 2 is one of the simplest special cases. As

another example, the case (15), written by (17) instead of by folding, would be

$$F_{j_1 \cdots j_{12}} = \Sigma \pm A^{(1)}_{i_1 i_2 j_1 j_2 j_3} A^{(2)}_{i_1 i_2 j_4 j_5 j_6} A^{(3)}_{i_1 i_2 j_7 j_8 j_9} A^{(4)}_{i_1 i_2 j_{10} j_{11} j_{12}} \qquad (18)$$

and similarly for all cases of $\mathbf{F}_k((\mathbf{A}^n))$.

When $p-k$ is even, the dummy is non-signant, and no component of $\mathbf{F}_k((\mathbf{A}^n))$ vanishes indentically. But if this integer is odd the dummy is signant and any component vanishes identically for which one set of k indices $j_{ik+1} \cdots j_{ik+k}$ and another set with different i agree in value each to each; for if so two layers in a signant direction are alike. To take a simple case of this sort, let $n=2$, $p=3$, $k=2$, *i.e.*, fold the binary triadic $A_{i_1 i_2 i_3}$ into itself on the first factor only; $\mathbf{F}_2((\mathbf{A}^2))$ is then equal to the m.i.p. development of the 2-way determinant

$$\begin{vmatrix} e_{\alpha_1} e_{\alpha_2} A_{1\alpha_1\alpha_2} & e_{\alpha_1} e_{\alpha_2} A_{2\alpha_1\alpha_2} \\ e_{\beta_1} e_{\beta_2} A_{1\beta_1\beta_2} & e_{\beta_1} e_{\beta_2} A_{2\beta_1\beta_2} \end{vmatrix} \qquad (19)$$

which in matrix language gives

$$F_{j_1 j_2 j_3 j_4} = A_{1\, j_1 j_3} A_{2 j_2 j_4} - A_{2\, j_1 j_3} A_{1 j_2 j_4}; \qquad (20)$$

the indices j_1 and j_2 belong to i_2, and j_3 and j_4 belong to i_3. Four of the sixteen components vanish. The 12 other components are numerically alike in pairs of opposite sign. Thus there are only six distinct components, neglecting signs and null components. These six are in fact the 2d order minors of the first index (V, p. 170) or, what is the same thing, the duplex minors of the other two indices (V, p. 173). The way these minors and their negatives occur in the matrix $F_{j_1 j_2 j_3 j_4}$ will appear from the following four-way display of the components (20)

$$\begin{array}{cc|cc} 0 & F_{1113} & F_{1211} & F_{1213} \\ -F_{1113} & 0 & F_{1221} & F_{1223} \\ \hline -F_{1211} & -F_{1221} & 0 & F_{2213} \\ -F_{1213} & -F_{1223} & -F_{2213} & 0 \end{array} \qquad (21)$$

in which the components taken as positive are the determinants of the matrix V, p. 170 (10).

Similarly for any values of n and p, if we take $k=p-1$, that is if we fold $[\mathbf{A}^n]$ on the first index only, we determine a matrix $\mathbf{F}_{p-1}((\mathbf{A}^n))$ some of whose components vanish while the rest are the n^{th}-order simple minors of the first index, each occurring $n!$ times with proper sign.

8. **Extension of the concept of minor.**

It has thus been shown that the operation $\mathbf{F}_k((\mathbf{A}^n))$ performed upon $A_{i_1..i_p}$ yields a multiply covariant matrix of class kn whose components $F_{j_1..j_{kn}}$ are the $(p-k+1)$-way determinants (17). By analogy it appears natural to regard these determinants as minors of $A_{i_1..i_p}$ in a slightly extended sense. Varying the first $p-k$ indices from 1 to n while the remaining k indices are fixed in value yields a $(p-k)$-way block of n^{p-k} components. There are n^k choices of the fixed indices. The n^k blocks, each of $(p-k)$ dimensions, if adjacent to one another, form a matrix of $p-k+1$ dimensions or ways. The files in all but one direction will contain n components each; in this one direction each file will contain n^k components, — these latter files being in fact identical with the k-plex files of the last k indices (V, p. 174). The determinants (17) are then merely the $(p-k+1)$-way n^{th}-order determinants of this matrix (with proper sign) under the following fundamentally important proviso: *repetitions are to be allowed in using the $(p-k)$-way blocks*. All directions in any $(p-k)$-way block are signant. The direction of the multiplex files (corresponding to the dummy in (17)) is signant or not according as $p-k+1$ is even or odd. These determinants I shall call the *$(p-k+1)$-way n^{th}-order simple minors on the first $p-k$ indices*. They include as special cases the two-way n^{th}-order simple minors on the first index defined in V, p. 169.

As examples of the matrix from which these minors are formed, the case (20) corresponds to the two-way matrix of V, p. 170, as already noted. The case (6) corresponds to the unmodified cubic matrix of the $A_{i_1 i_2 i_3}$, the repetitions occurring in the layers of the third direction. The case (18) corresponds to a three-way

matrix of $4\times4\times64$ components, with repetitions allowed in the 4×4 layers.

9. **Multiple concomitants in general.**

With the notation of Art. 2, if $\mathbf{F}'$ is obtained by letting all indices of $\mathbf{F}$ be contragredient to the indices of $\mathbf{A}$ with which they are associated, instead of being cogredient as originally defined, and if $\overline{\mathbf{F}}=\Delta^{\mu}\mathbf{F}'$, I shall say that $\mathbf{F}$ is a *multiple contravariant* of $\mathbf{A}$. More generally, if some indices of $\mathbf{F}$ are cogredient and some contragredient to their associated indices of $\mathbf{A}$, I shall say that $\mathbf{F}$ is a *multiple concomitant* of $\mathbf{A}$ when $\overline{\mathbf{F}}=\Delta^{\mu}\mathbf{F}'$.

The following generalization of Theorem V may now be given:

Theorem VI. The polyadics $\mathbf{F}_k((\mathbf{A}^q))$ are multiple concomitants of $\mathbf{A}$.

Proof. By definition the required polyadics are obtained from $[\mathbf{A}^q]$ by folding all but the last k factors. It will be sufficient to carry out the case $n=4$, $p=5$, $k=3$, $q=2$, the proof being in all cases similar. We have symbolically

$$\mathbf{F}_3((\mathbf{A}^2))=(\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)\mathbf{a}_3\mathbf{b}_3\mathbf{a}_4\mathbf{b}_4\mathbf{a}_5\mathbf{b}_5 \tag{22}$$

where $(\mathbf{a}_1\mathbf{b}_1)$ denotes the space complement of $\mathbf{a}_1\mathbf{b}_1$ and similarly in all cases. The indices not folded are evidently multiply covariant since for them $\overline{\mathbf{F}}=\mathbf{F}'$ with $\mu=0$. If $i_m=i_1$ we have $(\mathbf{a}'_1\mathbf{b}'_1)=\Delta(\mathbf{a}_1\mathbf{b}_1)''$ by IV, p. 252, and similarly in all cases where less than n vectors are folded; that is $\overline{\mathbf{F}}=\Delta\mathbf{F}'$ for any folded index which is contragredient in $\mathbf{F}$. Therefore $\mathbf{F}$ is multiply contravariant on those of its indices which belong to a folded index of $\mathbf{A}$.

Corollary. The polyadics $\mathbf{F}_0((\mathbf{A}^q))$ are multiple contravariants of $\mathbf{A}$ when q is less than n.

10. **Properties of the multiple concomitants $\mathbf{F}_k((\mathbf{A}^q))$.**

When $q<n$ there are no vacant indices. The results of III, pp. 228–231 hold without change. The components of $\mathbf{F}_k((\mathbf{A}^q))$ are in general determinants of class $p-k+1$ and order q which may be expressed in a form closely resembling (17).

$$\Sigma\pm\mathrm{A}^{(1)}_{i_1}\cdots i_{p-k}r_1\cdots z_1\cdots\mathrm{A}^{(q)}_{i_1}\cdots i_{p-k}r_q\cdots z_q \tag{23}$$

where the kq indices $r_i \cdots z_i$ $(i=1, \cdots, q)$ have fixed values and where the upper numbers $(1) \cdots (q)$ denote q distinct choices of the values of the indices $i_1 \ldots i_{p-k}$, which are signant. The dummy upper index is signant only when $p-k+1$ is even. The important difference, as compared with (17), is that $i_1 \cdots i_{p-k}$ instead of running from 1 to n are each restricted to a range of q particular values chosen from $1, 2, \cdots, n$. Each component of the form (23) occurs not once merely but is the coefficient of the polyadic (38) of III, p. 231, as very fully exemplified in that paper.

As a further illustration, any non-vanishing component of (22) is $\pm$ a cubic determinant of the second order

$$\Sigma \pm A_{i_1 i_2 r_1 s_1 t_1}^{(1)} A_{i_1 i_2 r_2 s_2 t_2}^{(2)} \tag{24}$$

in which the index i_1 is restricted to two of the four values 1, 2, 3, 4, while i_2 is also restricted to two values, which may be the same as the range of i_1 or different. For instance let i_1 have the range 1, 2 and i_2 have the range 1, 3 and let $r_1, s_1, t_1, r_2, s_2, t_2$ have, in order, the values 1, 2, 3, 2, 3, 4: certain components of (22) will be $\pm$ the cubic determinant

$$\begin{vmatrix} A_{11123} & A_{13123} & A_{11234} & A_{13234} \\ A_{21123} & A_{23123} & A_{21234} & A_{23234} \end{vmatrix} \tag{25}$$

in which the first two subscripts are signant.

As a generalization of a similar remark at the close of Art. 7 we may now note that if, for any values of n, p, and q, we take $k=p-1$, that is if we fold $[\mathbf{A}^q]$ on the first index only, we determine a matrix $\mathbf{F}_{p-1}$ $((\mathbf{A}^q))$ some of whose components vanish, while the rest are the q^{th}-order simple minors of the first index, each occurring a certain number of times with proper sign.

11. Further extension of the concept of minor.

It has thus been shown that the operation $\mathbf{F}_k((\mathbf{A}^q))$ performed upon the given matrix $A_{i_1 \ldots i_p}$ yields a new matrix whose components are either zero or else one of the $(p-k+1)$-way q^{th}-order determinants (23) with proper sign. Considering again the matrix of $p-k+1$ ways built of $(p-k)$-way blocks as described in Art. 8, the determinants (23) are merely q^{th}-order determinants

of this matrix with repetitions allowed as before, and may be called the *$(p-k+1)$-way q^{th}-order simple minors of the first $p-k$ indices.*

For example (25) is a cubic 2d-order simple minor of $A_{i_1 i_2 i_3 i_4 i_5 i_6}$ on $i_1 i_2$, which may be picked from the three-way matrix of $4\times4\times64$ components mentioned at the close of Art. 8.

12. **Folding on arbitrarily chosen indices.**

More generally we may fold on *any* $p-k$ indices instead of merely on the *first* $p-k$. The notation $\mathbf{F}_k((\mathbf{A}^q))$ temporarily adopted in III, p. 228 now requires amplification. Since the formation of the m.i.p. product $[\mathbf{A}^q]$ in any case precedes folding, the folded indices may be indicated by superscripts appended to $[\mathbf{A}^q]$; or those not folded may be written as subscripts; or both sets may be written. Thus the four expressions

$$\mathbf{F}_1((\mathbf{A}^2)) \qquad [\mathbf{A}^2]^{i_1 i_2} \qquad [\mathbf{A}^2]_{i_3} \qquad [\mathbf{A}]^{i_1 i_2}_{i_3}$$

will all be ways of denoting the matrix (6) when $n=2$, $p=3$. There can exist no one notation most convenient in every circumstance. I shall continue to reserve $\mathbf{F}_k((\mathbf{A}^q))$ for indicating folding on all but the last k indices.

In what follows I shall reserve small capitals A, B, $\cdots$, G to denote $p-k$ indices chosen from $i_1 \cdots i_p$ on which folding occurs, and $r, s, \cdots, z$ to denote the remaining k indices on which folding does not occur: this is the notation of III, p. 231. The expression

$$[\mathbf{A}^q]^{\mathrm{A}\cdots\mathrm{G}}_{r\cdots z} \tag{26}$$

accordingly means the new matrix obtained from $A_{i_1 \cdots i_p}$ by first forming the m.i.p. product $[\mathbf{A}^q]$ and then replacing each group of vector factors carrying the designated indices A $\cdots$ G by its space complement.

There is nothing in the proof of Theorem VI requiring folding to be on the *first* p−k indices. It is evident therefore that (26) is a multiple concomitant of **A**. Its components can be written as in (23), replacing $i_1 \cdots i_{p-k}$ by A, $\cdots$, G,

$$\Sigma \pm A^{(1)}_{\mathrm{A}\cdots\mathrm{G}\, r_1 \cdots z_1} \cdots A^{(q)}_{\mathrm{A}\cdots\mathrm{G}\, r_q \cdots z_q} \tag{27}$$

the upper numbers $(1)\cdots(q)$ denoting q choices of values for the indices $A\cdots G$ which are signant and which are restricted to a range of q values. It is to be emphasized that the indices here denote any desired permutation of $i_1\cdots i_p$; but without loss of generality we may usually suppose the order of $A\cdots G$ to be that of the same indices in $i_1\cdots i_p$; and similarly for $r\cdots z$.

In any chosen component (27) the q sets of k indices each $r_i\cdots z_i$ $(i=1, 2, \cdots q)$ have fixed values: those particular components wherein these q sets of values occur in natural order (V, p. 169) may be called the $(p-k+1)$-way q^{th} *-order simple minors of* $A_{i_1..i_p}$ *on the indices* A, B, $\cdots$, G. All other components (27) either vanish or differ from these minors at most in sign.

The foregoing facts may be stated as

Theorem VII. Any polyadic $[\mathbf{A}^q]^{A\cdots G}$ is a multiple concomitant of $\mathbf{A}$ of which any component is either zero or is $\pm$ a $(p-k+1)$-way q^{th}-order simple minor of $A_{i_1..i_p}$ on the indices $A\cdots G$; and all such minors occur at least once among the components of this polyadic.

13. **Extension of the concept of simple rank.**

I shall now lay down the following

Definition II. The $(p-k+1)$-way simple rank of $A_{i_1..i_p}$ on the $p-k$ indices $A\cdots G$ is the greatest value of q for which the $(p-k+1)$-way q^{th}-order simple minors on these indices are not all zero.

Rank as thus defined evidently includes as a special case simple (two-way) rank on a single index (V, p. 170), which, in turn, is a generalization of the ordinary rank of a two-way matrix.

Theorem IV and its two corollaries hold without verbal change when $\mathbf{F}$, instead of being merely a multiple covariant, is any multiple concomitant.

Theorem VIII. The $(p-k+1)$-way simple rank of $A_{i_1..i_p}$ on any $p-k$ designated indices A, B, $\cdots$ G is a multiply invariant number belonging to $\mathbf{A}$.

Proof. Let the rank in question be a number q_1. Suppose $q>q_1$ and let $\mathbf{F}=[\mathbf{A}^q]^{A\cdots G}$. By Theorem VII' we have $\mathbf{F}=0$.

By Corollary II of Theorem IV we therefore have $\overline{\mathbf{F}}=0$; that is, q_1 is a multiple invariant of $\mathbf{A}$.

It is obvious that any $(p-k+1)$-way simple rank as thus defined is in general and at most equal to n.

14. **Explicit formulation of the matrix $[\mathbf{A}^q]^{A\cdots G}$.**

How minors of $A_{i_1}..i_p$, their negatives, and the zero components, occur in the matrix $\mathbf{F}_k((\mathbf{A}^q))$ is fully determined by the polyadic (38) of III, p. 231, and the ease with which any desired component may be found was illustrated in detail in that place. When the folding is on *any* $p-k$ indices we may suppose an isomer of $\mathbf{A}$ to be first formed such that its first $p-k$ indices are the ones to be folded. The method of III then applies to this isomer, yielding a polyadic similar to III, (38). The order of indices $a\cdots gr\cdots z$ of that polyadic may then be rearranged to agree with the original order $i_1\cdots i_p$. We thus pass from the polyadic form of $[\mathbf{A}^q]^{A\cdots G}$ to the explicit formulation of any or all components.

If preferred, however, this multiple concomitant can be defined without use of the folding or symbolic process. Let the determinantal polyadic

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{vmatrix} \tag{28}$$

be denoted by $\mathbf{E}$. Its components $E_{j_1}..j_n$ are $+1$ or -1 when the indices are an even or odd permutation of $1, 2, \cdots, n$ and zero when two indices are equal. The space complement of q vectors $a_{1,A_1}\cdots a_{q,A_q}$ is then

$$E_{j_1}..j_{n-q}{}_{A_1}..{}_{A_q}a_{1,A_1}\cdots a_{q,A_q} \tag{29}$$

summed on $A_1\cdots A_q$. It follows at once from the definition of $\mathbf{F}_k((\mathbf{A}^q))$ that this matrix is the same as

$$\begin{aligned} &F_{a_1}..a_{n-q}b_1..b_{n-q}\cdots g_1..g_{n-q}r_1..r_qs_1..s_q\cdots z_1..z_q \\ &= E_{a_1}..a_{n-q}{}_{A_1}..{}_{A_q}\cdots E_{g_1}..g_{n-q}{}_{G_1}..{}_{G_q}A_{A_1}..{}_{G_1}r_1..z_1\cdots A_{A_q}..{}_{G_q}r_q..z_q \end{aligned} \tag{30}$$

summed on the $q(p-k)$ indices $A_1 \cdots G_1 \cdots A_q \cdots G_q$. If (30) denotes $\mathbf{F}_k((\mathbf{A}^q))$, *i.e.*, if the *first* $p-k$ indices alone are folded the order of indices $A \cdots G$ is the same as $i_1 \cdots i_{p-k}$. Their order agrees with that of the groups of complementary indices $a, \cdots g$. The indices $r \cdots z$ are the same as $i_{p-k+1} \cdots i_p$. But (30) may equally well denote $[\mathbf{A}^q]^{A \cdots G}$, *i.e.*, the folding may be on *any* $p-k$ indices $A \cdots G$, provided we are willing to suppose that the indices $A \cdots Gr \cdots z$ stand, not in the order written, but as they actually occur in $i_1 \cdots i_p$. The order of the groups $a \cdots g$ is always the same as that of $A \cdots G$.

There is still another method for passing from symbolic to explicit formulation, closely analogous with one used in simple cases in the classical theory of invariants, namely, given such a symbolic form as (22), expand all vectors in terms of the $\mathbf{e}_1 \cdots \mathbf{e}_n$ and multiply out. After the multiplication, any product such as $a_{1,r}\, a_{2,s}\, a_{3,t}\, a_{4,u}\, a_{5,v}$ is replaced by A_{rstuv}, and any product of b's similarly is replaced by A with corresponding subscripts, just as in the usual symbolic method.

Just as in the classical theory, however, it will usually be true that any explicit formulation is less perspicuous both for demonstration of theorems and for display of results than the symbolic method, which, in the present investigation, becomes the polyadic method, — or rather one aspect of the polyadic method. It is also true that ordered determinants afford a method in a sense intermediate between a fully explicit display of components and one purely symbolic.

15. Extension of the concept of multiplex rank.

The definition of rank laid down in Art. 13 suffers from a fairly obvious and quite unnecessary restriction. In building a matrix out of n^k blocks of n^{p-k} components each (Art. 8) it is not necessary that these blocks be of order n in all directions; they do not need to be hyper-cubical, *i.e.*, with n components in every file. Each block of n^{p-k} components may equally well have less than $p-k$ directions, containing, therefore, in one or more directions, files of n^t components, where t is an integer $\leqq p-k$, not necessarily the same integer for all ways in the block. In other words the $p-k$ indices of $A_{i_1 \cdots i_p}$ which, being allowed to run from *1* to n, the

other indices being fixed, yield the n^{p-k} components occurring in the block, may be collected into $\pi-k$ groups of $t_1, t_2, \cdots, t_{\pi-k}$ indices each; the indices of each group are then replaced by a multipartite index as in V, p. 174. As the t_i indices of any group run, in natural order, from *1* to n, the corresponding multipartite index u_i will run from *1* to n^{t_i}. We thus have identically

$$A_{i_1 \cdots i_p} = A_{u_1 \cdots u_{\pi-k} r \cdots z} \tag{31}$$

where, as before, $r \cdots z$ are the k indices fixed in value over any chosen block. There will thus be n^k blocks as before, which, if adjacent to one another in natural order, will form a matrix of $n^{t_1} \times n^{t_2} \cdots \times n^{t_{\pi-k}} \times n^k$ components. The q^{th}-order determinants which can be picked from this matrix, allowing repetitions of blocks as before, may be called the $(\pi-k+1)$-*way* q^{th}-*order multiplex minors on the* $\pi-k$ *designated multipartite indices*. These minors will have precisely the form (27) with this difference of interpretation merely: the indices A $\cdots$ G now are the same as the multipartite indices $u_1 \cdots u_{\pi-k}$ instead of being the same as a selected group of $p-k$ simple indices.

Definition III. The $(\pi-k+1)$-way multiplex rank of $A_{i_1 \cdots i_p}$ on $\pi-k$ simple or multipartite indices A, B, $\cdots$ G is the greatest value of q for which the $(\pi-k+1)$-way q^{th}-order multiplex minors on these indices are not all zero.

Rank as thus defined evidently includes as a special case the t-plex rank on one group of t indices as defined V, p. 174. It also includes simple rank of definition II when $\pi = p$, *i.e.*, when all indices are simple.

16. Multiplex space complement.

It is now necessary to show that generalized rank as above defined is, in any case, a multiply invariant number. This I shall do by means of a (purely algebraic) extension of the space complement of Dr. Naess, (II, p. 179). Let there be q matrices or polyadics all of class t and carrying the same indices, thus: $a_{i_1 \cdots i_t}, b_{i_1 \cdots i_t}, \cdots, g_{i_1 \cdots i_t}$. Now let the t simple indices be replaced by one t-partite index u running in natural order from *1* to n^t. The polyadics, in other words, are to be treated as vectors

in space of N dimensions, where $N=n^t$. Let the t-ads $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_t}$ be denoted by $\mathbf{E}_u$. I now define the *multiplex space complement* of the q polyadics to be a square determinant of order N having the N fundamental t-ads $\mathbf{E}_1\cdots\mathbf{E}_N$ for the elements of its first $N-q$ (identical) rows, and having the components of the polyadics for its last q rows

$$\begin{vmatrix} \mathbf{E}_1 & \cdots & \mathbf{E}_N \\ \cdots & \cdots & \cdots \\ \mathbf{E}_1 & \cdots & \mathbf{E}_N \\ a_1 & \cdots & a_N \\ \cdots & \cdots & \cdots \\ g_1 & \cdots & g_N \end{vmatrix} \tag{32}$$

it being understood that the expansion of the determinant is by m.i.p. (III, p. 212), in the order of rows.

The simplest possible case of a multiplex space complement is given by $n=t=2$. If also $q=2$, so that we have two dyadics $a_{i_1i_2}$, and $b_{i_1i_2}$, their multiplex space complement will be

$$\begin{vmatrix} \mathbf{e}_1\mathbf{e}_1 & \mathbf{e}_1\mathbf{e}_2 & \mathbf{e}_2\mathbf{e}_1 & \mathbf{e}_2\mathbf{e}_2 \\ \mathbf{e}_1\mathbf{e}_1 & \mathbf{e}_1\mathbf{e}_2 & \mathbf{e}_2\mathbf{e}_1 & \mathbf{e}_2\mathbf{e}_2 \\ a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \end{vmatrix} \tag{33}$$

with m.i.p. development, by rows.

Theorem IX. The multiplex-space complement of q polyadics is a multiple contravariant of these polyadics when $q<t^n$ and the corresponding indices or vector factors of the polyadics are cogredient each to each.

Proof. It will be sufficient to carry out the proof on (33). Calling this determinant $\mathbf{F}$ and taking $i_m=i_1$ we have, by definition,

$$\bar{\mathbf{F}}=\begin{vmatrix} \mathbf{f}_1\mathbf{e}_1 & \mathbf{f}_1\mathbf{e}_2 & \mathbf{f}_2\mathbf{e}_1 & \mathbf{f}_2\mathbf{e}_2 \\ \mathbf{f}_1\mathbf{e}_1 & \mathbf{f}_1\mathbf{e}_2 & \mathbf{f}_2\mathbf{e}_1 & \mathbf{f}_2\mathbf{e}_2 \\ a_{r1}T_{r1} & a_{r2}T_{r1} & a_{r1}T_{r2} & a_{r2}T_{r2} \\ b_{r1}T_{r1} & b_{r2}T_{r1} & b_{r1}T_{r2} & b_{r2}T_{r2} \end{vmatrix} \tag{34}$$

where $\mathbf{f}_1$, $\mathbf{f}_2$ are the new basis vectors of the first index and in the last two rows we sum for $r=1, 2$. On the other hand let $\overline{T}_{rs}$ be the components of the contragredient transformation to be applied to the basis vectors of the first index in (33) so that

$$\mathbf{F}'=\begin{vmatrix} \mathbf{f}_s\mathbf{e}_1\overline{T}_{1s} & \mathbf{f}_s\mathbf{e}_2\overline{T}_{1s} & \mathbf{f}_s\mathbf{e}_1\overline{T}_{2s} & \mathbf{f}_s\mathbf{e}_2\overline{T}_{2s} \\ \mathbf{f}_s\mathbf{e}_1\overline{T}_{1s} & \mathbf{f}_s\mathbf{e}_2\overline{T}_{1s} & \mathbf{f}_s\mathbf{e}_1\overline{T}_{2s} & \mathbf{f}_s\mathbf{e}_2\overline{T}_{2s} \\ a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \end{vmatrix} \tag{35}$$

To see that $\overline{\mathbf{F}}=\Delta^2\mathbf{F}'$ it is merely necessary to note the identity

$$\Delta^2=\begin{vmatrix} T_{11} & 0 & T_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ T_{21} & 0 & T_{22} & 0 \\ 0 & T_{21} & 0 & T_{22} \end{vmatrix} \tag{36}$$

whence multiplying (36) into (35) gives (34) by virtue of the identity $\mathbf{f}_s\overline{T}_{rs}T_{ri}=\mathbf{f}_i$. It has thus been shown that (33) is a contravariant of $a_{i_1i_2}$ and $b_{i_1i_2}$ on the first index. The proof is in all cases similar.

When no confusion results I shall denote the multiplex space complement of several polyadics $\mathbf{A}, \mathbf{B}, \cdots, \mathbf{G}$ by $(\mathbf{A}, \mathbf{B}, \cdots, \mathbf{G})$, which reduces to the space complement of Dr. Naess when these polyadics are vectors. When these polyadics are all transformed on the same index i_m the multiplex space complement of the transformed polyadics may be denoted by $(\mathbf{A}', \mathbf{B}', \cdots, \mathbf{G}')$. On the other hand when the multiplex space complement $(\mathbf{A}, \mathbf{B}, \cdots, \mathbf{G})$ is contragrediently transformed on the indices which belong to i_m the result may be written $(\mathbf{A}, \mathbf{B}, \cdots, \mathbf{G})''$. Theorem IX is thus the same as

$$(\mathbf{A}', \mathbf{B}', \cdots, \mathbf{G}')=\Delta^n(\mathbf{A}, \mathbf{B}, \cdots, \mathbf{G})'' \tag{37}$$

a generalization of IV, p. 252.

If we have n^t polyadics their multiplex space complement is evidently a scalar, which, by a similar proof, is a multiple inva-

riant of these polyadics, assuming them to carry the same (*i.e.*, cogredient) indices. For example if we have four binary dyadics the determinant of their 16 components is a multiple invariant. If the four dyadics are merely four dyads $\mathbf{a}_1\mathbf{a}_2$, $\mathbf{b}_1\mathbf{b}_2$, $\mathbf{c}_1\mathbf{c}_2$, and $\mathbf{d}_1\mathbf{d}_2$, still with $n=2$, their multiplex space complement $(\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2, \mathbf{c}_1\mathbf{c}_2, \mathbf{d}_1\mathbf{d}_2)$ is by definition

$$\begin{vmatrix} a_{1,\,1}a_{2,\,1} & a_{1,\,1}a_{2,\,2} & a_{1,\,2}a_{2,\,1} & a_{1,\,2}a_{2,\,2} \\ b_{1,\,1}b_{2,\,1} & b_{1,\,1}b_{2,\,2} & b_{1,\,2}b_{2,\,1} & b_{1,\,2}b_{2,\,2} \\ c_{1,\,1}c_{2,\,1} & c_{1,\,1}c_{2,\,2} & c_{1,\,2}c_{2,\,1} & c_{1,\,2}c_{2,\,2} \\ d_{1,\,1}d_{2,\,1} & d_{1,\,1}d_{2,\,2} & d_{1,\,2}d_{2,\,1} & d_{1,\,2}d_{2,\,2} \end{vmatrix} \tag{38}$$

and, since a dyadic is a sum of dyads, $(\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2, \mathbf{c}_1\mathbf{c}_2, \mathbf{d}_1\mathbf{d}_2)$ is symbolic for $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ when $t=2$. Since (38) is a multiple invariant it is an invariant, hence expressible in terms of the 12 simple determinants $(\mathbf{a}_1\mathbf{b}_1)$, $(\mathbf{a}_1\mathbf{c}_1)$, $\cdots$, $(\mathbf{c}_1\mathbf{d}_1)$, $(\mathbf{a}_2\mathbf{b}_2)$, $\cdots$, $(\mathbf{c}_2\mathbf{d}_2)$, (Cf. Weitzenbock, Invariantenth. s. 258) because determinants of factors with unlike indices, *e.g.*, $(\mathbf{a}_1\mathbf{b}_2)$ would destroy the multiple character. This is one of the simplest examples of the fact that multiplex folding can be expressed in terms of simple folding, — a transformation which, for multiplex space complements in general, involves the determinant $\mathbf{E}$ as well as the simple space complements of given vector factors. (See Examples 7–12 at end of paper.)

More generally, if we have q polyadics which are merely t-ads, $\mathbf{a}_1\mathbf{a}_2 \cdot\cdot \mathbf{a}_t$, $\cdots$, $\mathbf{g}_1\mathbf{g}_2 \cdot\cdot \mathbf{g}_t$, their multiplex space complement

$$(\mathbf{a}_1\mathbf{a}_2 \cdot\cdot \mathbf{a}_t,\ \mathbf{b}_1\mathbf{b}_2 \cdot\cdot \mathbf{b}_t,\ \cdots,\ \mathbf{g}_1\mathbf{g}_2 \cdot\cdot \mathbf{g}_t) \tag{39}$$

is symbolic for the multiplex space complement of q general polyadics $(\mathbf{A}, \mathbf{B}, \cdot\cdot, \mathbf{G})$. If $q=n^t$ then (39) is a determinant whose rows are the components of the t-ads in natural order, the first row being

$$a_{1,\,11\cdot\cdot1}a_{2,\,11\cdot\cdot1} \cdots a_{t,\,11\cdot\cdot1},\quad a_{1,\,11\cdot\cdot1}a_{2,\,11\cdot\cdot1} \cdots a_{t,\,11\cdot\cdot n} \quad \cdots \text{etc.},$$

the last element of the first row being $a_{1,\,nn\cdot\cdot n}a_{2,\,nn\cdot\cdot n} \cdots a_{t,\,nn\cdot\cdot n}$ and the other rows being similar in the b's, c's, $\cdots$, g's. This determinant, being an invariant, is expressible in terms of the

simple determinants ($\mathbf{a}_1\mathbf{b}_1 \cdots$) each containing n vector factors of like index.

17. **Multiplex folding.**

A general definition of multiplex folding can now be given. Let a matrix or polyadic $A_{i_1 \cdots i_p}$ or $\mathbf{A}$ be reshaped as in Art. 15, *i.e.*, we replace $p-k$ simple indices by $\pi-k$ simple or multipartite indices. We next form the m.i.p. product $[\mathbf{A}^q]$ just as in simple folding. If one of the indices to be folded is still a simple index, we replace any group of q vector factors of that index by their simple space complement. If t_i simple indices have been replaced by a t_i-partite index the qt_i vector factors bearing these t_i simple indices are replaced, in any term of $[\mathbf{A}^q]$, by the multiplex space complement of the q t-ads whose m.i.p. product constitutes the group in question. After all such replacements have been made the vector factors of the various indices are to stand in the original order $i_1 \cdots i_p$. The resulting polyadic is then fully determined. More generally we may treat any q polyadics carrying the same indices in this way. The result will be denoted by

$$[\mathbf{A}, \mathbf{B}, \cdots \mathbf{G}]^{\mathfrak{AB}\cdots} \tag{40}$$

where the superscripts denote the various groups of simple indices which are replaced by multipartite indices, some of which may, of course, still consist of a single index.

The number of indices grouped into a multipartite index being t_i, let λ be the least of the numbers t_i and let γ be the greatest. It is evident that, in multiplex folding, q does not exceed n^λ.

If $q=n^\lambda$ one at least of the multiplex space complements in any term of (40) is a scalar, for, as above pointed out, (39) is a scalar when $q=n^t$. The indices of the group will be vacant, and (40) will be multiply invariant on these indices because (39) is then a multiple invariant of the λ-ads which enter.

Any group for which $q<n^t$ yields, in any term of (40), a polyad of $t(n^t-q)$ indices corresponding to t indices of $A_{i_1 \cdots i_p}$ each of which is (n^t-q)-filled. By Theorem IX, together with the distributively invariant character of all steps of the processes concerned, (40) will be multiply contravariant on this group of t indices.

Finally, (40) will be multiply covariant on the k indices not folded.

It has thus been shown that (40) is a multiple concomitant of the given polyadics **A**, **B**, $\cdots$, **G**, assumed to carry the same indices. It remains to show that, when these polyadics are all equal to **A**, the components of (40) are either zero or else the multiplex minors of Art. 15, with proper sign, and that each minor occurs one or more times.

If all the numbers t_i were equal, the whole of III, Art. 11 could now be repeated without change, except that the folded indices would be multipartite and $\pi-k$ in number instead of $p-k$; N $(=n^t)$ would replace n; and fundamental t-ads would serve as N basis vectors. In this case, therefore, the components of $[\mathbf{A}^q]^{A\cdots G}$ would have precisely the form (27) and the required result would be established.

The numbers t_i being in general different I shall suppose the matrix $A_{u_1\cdots u_{\pi-k}r\cdots z}$ of (31) to be extended by bordering with fictitious components so that each of the π indices runs from 1 to n^γ. The true matrix is then a special case of the fictitious matrix such that the multiplex rank on any index u_i happens to be n^{t_i} (or less) instead of n^γ, and the simple rank on any one of the indices $r\cdots z$ happens to be not greater than n.

To describe this fictitious matrix in another way, we know that, for sufficiently large h, we can always develop **A** in the form (3). All t_i-ad factors B_{ij,u_i} $(j=1, 2, \cdots, h)$ are linearly expressible in terms of R_i of themselves where R_i is the rank on u_i; such a suitably chosen set of R_i polyadics may be taken as basis vectors of the index u_i, for the whole notion of multiple invariance rests on the fact that the basis vectors of a given index have nothing to do with those of any other index but are merely so many linearly independent quantities not commutative in multiplication. Now since R_i is certainly not greater than n^γ we could express all B_{ij,u_i} in terms of a suitably chosen set of n^γ polyadics or basis vectors if so many linearly independent ones existed which is in general not the case. However, we introduce enough fictitious basis vectors of the index u_i to fill the gap.

Let the reasoning of III, Art. 11 be now carried out on this partly fictitious matrix or polyadic of order n^γ. The components

of a fictitious $[\mathbf{A}^q]^{A\cdots G}$ are thus shown to have the form III, (39) which is the same as (27) of this paper, differing from (27) only in the supposition that we have n^γ values from which to select the range of q values for any index $A \cdots G$ and the fixed values of $r \cdots z$. I shall now show that the introduction of fictitious components and fictitious basis vectors of the π indices $u_1 \cdots u_{\pi-k} r \cdots z$ cannot alter the form of the scalar components of $[\mathbf{A}^q]^{A\cdots G}$.

This follows immediately on consideration of a typical term in the polyadic expansion by m.i.p. of $[\mathbf{A}^q]$ which, by folding on the π partly fictitious indices yields the fictitious $[\mathbf{A}^q]^{A\cdots G}$. Such a typical term, after folding, is the symbolic expression for the fictitious $[\mathbf{A}^q]^{A\cdots G}$. Folding on a partly fictitious index means constructing a fictitious space complement in an n^γ-space instead of an n^{δ_i}-space or merely an n-space. But this fictitious space complement has precisely the same non-vanishing scalar components as the real space complement, each component being multiplied by a fictitious polyadic. For example if actual folding produced (33), the duplex space complement of two binary dyadics, and if $\gamma = 3$, the corresponding fictitious space complement would be a determinant of the eighth order whose first six (identical) rows are

$$\mathbf{e}_1\mathbf{e}_1 \quad \mathbf{e}_1\mathbf{e}_2 \quad \mathbf{e}_2\mathbf{e}_1 \quad \mathbf{e}_2\mathbf{e}_2 \quad \mathbf{E}_5 \quad \mathbf{E}_6 \quad \mathbf{E}_7 \quad \mathbf{E}_8$$

where the last four elements are fictitious; and the seventh row would be

$$a_{11} \quad a_{12} \quad a_{21} \quad a_{22} \quad a_5 \quad a_6 \quad a_7 \quad a_8$$

and the eighth row similarly in the b's. Since a_5, a_6, a_7, a_8, b_5, b_6, b_7 and b_8 are actually zero, the scalar components of this fictitious space complement are, by a Laplacean development, the same as those of (33), and similarly in all cases. Therefore the symbolic expression for $[\mathbf{A}^q]^{A\cdots G}$, on multiplying out the folded groups, has the same scalar components in the fictitious as in the actual case, neglecting signs and zero components. Passing from symbolic to explicit form in the usual manner, the same holds for the complete development. Thus the reasoning of III, Art. 11, while it replaces III, (38) by a fictitious polyadic, does not alter the form III, (39) of the components, nor the form (27) of this paper. We have therefore proved

Theorem X. A polyadic $[\mathbf{A}^q]^{A\cdots G}$ where $A \cdots G$ are $\pi-k$ multipartite (or simple) indices is a multiple concomitant of $A_{i_1 \cdots i_p}$ of which any component is either zero or is $\pm$ one of the $(\pi-k+1)$-way q^{th}-order minors (in general multiplex) of $\mathbf{A}$; and all such minors occur at least once among the components of this multiple concomitant.

Hence also

Theorem XI. The $(\pi-k+1)$-way multiplex rank of $A_{i_1 \cdots i_p}$ on any $\pi-k$ designated multipartite indices is a multiply invariant number.

The proof is similar to that of Theorem VIII.

It may be remarked that any multiplex space complement, and hence $[\mathbf{A}^q]^{A\cdots G}$, can, if desired, be expressed in forms analogous to (29) and (30) by use of a multiplex analogue of (28). Such an explicit formulation will not be required in this paper, the polyadic or symbolic forms of these quantities being more adaptable to present purposes. See Example 13–17, page 78 below.

To give perhaps the simplest possible example of multiplex folding and related minors, take a binary tetradic, *i.e.*, $A_{i_1 i_2 i_3 i_4}$ with $n=2$. Keep i_1 simple, replace $i_2 i_3$ by a bipartite index, and leave i_4 not folded. Fold the tetradic into itself on the simple index i_1 and the bipartite index $i_2 i_3$, *i.e.*, make $[\mathbf{A}^2]^{i_1(i_2 i_3)}$ where the bipartite character of $i_2 i_3$ is indicated by parentheses. If we write the polyadic expansion of $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4 + \cdots \tag{41}$$

we shall have symbolically

$$[\mathbf{A}^2] = \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2\mathbf{a}_3\mathbf{b}_3\mathbf{a}_4\mathbf{b}_4 \tag{42}$$

and

$$[\mathbf{A}^2]^{i_1(i_2 i_3)} = (\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{a}_3,\ \mathbf{b}_2\mathbf{b}_3)\mathbf{a}_4\mathbf{b}_4 \tag{43}$$

with biplex folding on $i_2 i_3$ as defined and illustrated in Art. 16. The matrix of Art. 15 will consist of a pair of two-way blocks of which the first is

$$\begin{matrix} A_{1111} & A_{1121} & A_{1211} & A_{1221} \\ A_{2111} & A_{2121} & A_{2211} & A_{2221} \end{matrix} \tag{44}$$

and the second is obtained by advancing the fourth subscript. Since doubling of blocks is permitted there are 18 three-way 2d-order minors on the two indices i_1 and $(i_2 i_3)$. If the latter or bipartite index be called u and run from *1* to *4* when $(i_2 i_3)$ takes the bipartite values 11, 12, 21, and 22, these minors will be included in

$$\Sigma \pm \mathrm{A}_{i_1 ur}^{(t)} \mathrm{A}_{i_1 us}^{(\pi)} \tag{45}$$

where, in the manner of (27), the upper numbers indicate two choices of pairs of values for i_1 and u, while r and s are two values of i_4. Both i_1 and u are signant. For instance, if u has the range 2, 3, *i.e.*, 12 and 21, with $r=1$, $s=2$, we obtain the three-way minor

$$\begin{vmatrix} \mathrm{A}_{1121} & \mathrm{A}_{1211} \\ \mathrm{A}_{2121} & \mathrm{A}_{2211} \end{vmatrix} \begin{vmatrix} \mathrm{A}_{1122} & \mathrm{A}_{1212} \\ \mathrm{A}_{2122} & \mathrm{A}_{2212} \end{vmatrix} \tag{46}$$

The multiple concomitant (43) is a hexadic because the first space complement on the right is a scalar, the second is a tetradic, and $\mathbf{a}_4\mathbf{b}_4$ is a dyad. The first index of $\mathrm{A}_{i_1 i_2 i_3 i_4}$ is vacant, and the other three indices are two-filled.

18. **Rank of various grades.**

If we let $\sigma = \pi - k + 1$ any rank included in definition III may be indicated by R_σ followed by the designated indices in brackets. For example, if (43) vanishes in all components $\mathrm{R}_3[i_1(i_2 i_3)]$ is unity or zero, for the tetradic in question. Any rank included in definition III may be called a *primary* rank of $\mathrm{A}_{i_1 \cdots i_p}$. Evidently any matrix possesses a finite number of primary ranks.

These ranks have been shown to be related to multiple concomitants $[\mathbf{A}^q]^{AB\cdots}$. Any primary rank of one of these concomitants is, by Theorem IV, a multiple invariant of $\mathrm{A}_{i_1 i_2 \cdots i_p}$ and may be called a secondary rank of $\mathbf{A}$. Similarly we may have tertiary, and in general *plenary* ranks, according to grade. The ranks of any one grade are finite in number. These various ranks are not all independent.

Secondary ranks occurred in V although not so designated in that paper. In V, Art. 13, if the invariant I_a vanishes, $\mathrm{F}_1((\mathbf{A}^2))$ is of rank one, a secondary rank of $\mathbf{A}$, which becomes zero when

$F_1((\mathbf{A}^2))$ vanishes in all components. But in this latter case the three-way simple rank on i_1, i_2 is unity, illustrating that, as above noted, different ranks are not always independent.

Again, in V, Art. 18, the m_{abrs} are three-way minors on i_1, i_2. If any four of the nine conics are linearly related the duplex two-way rank of $F_1((\mathbf{A}^2))$ on its first two indices does not exceed three, a secondary rank of **A**. The Jacobians of these conics, in sets of three, are obtainable by folding $\mathbf{F}_1((\mathbf{A}^2))$ into itself, forming a *secondary concomitant* of **A**. The Hessians of these Jacobians in turn are obtainable by further folding, thus involving tertiary concomitants. In another paper I propose to inquire to what extent, and in what manner, the possibility of the general decomposition (3) of this paper is determined (for given h) when the plenary ranks of the matrix $A_{i_1 \cdots i_p}$ are known. But these plenary ranks are far from comprising all multiply invariant numbers.

19. **Further examples of multiple concomitants.**

The present paper makes no attempt to give a general theory of all multiple concomitants. I shall, however, add several examples of these quantities not included in $[\mathbf{A}^q]^{\lambda \cdots \sigma}$.

Example 1. Any isomer of **A** is a multiple covariant of **A**, provided the indices, after permutation, are cogredient with, *i.e.*, belong to, their original positions in **A**.

Example 2. With the same proviso, any isomer of a multiple covariant is a multiple covariant.

Example 3. The sum of two multiple covariants is a multiple covariant provided

1°. They are of the same number of indices.

2°. They are of the same degree in the $A_{i_1 \cdots i_p}$.

3°. Their corresponding indices have the same association with the indices $i_1 \cdots i_p$.

Example 4. The m.i.p. product of **A** into itself any number of times, namely $[\mathbf{A}^q]$ where q is an integer, is a multiple covariant of **A**.

Example 5. Given q vectors $\mathbf{a}, \mathbf{b}, \cdots, \mathbf{g}$, let the determinant having these vectors for elements of its q (identical) rows be denoted by $|\mathbf{a}, \mathbf{b}, \cdots, \mathbf{g}|$. In $[\mathbf{A}^q]$ let all groups of q vector factors carrying a chosen index be replaced by their determinant as just

defined (instead of by their space complement as in folding). The resulting polyadic is a multiple covariant whose components are either zero or $\pm$ the two-way q^{th} order simple minors on the chosen index, and each such minor occurs at least once.

Example 6. Let the process of Ex. 5 be called alternation. If we alternate on $p-k$ chosen indices the result is a multiple covariant whose components comprise the $(p-k+1)$-way q^{th}-order simple minors on the chosen indices. To illustrate, let $\mathbf{A}$ be a binary triadic and alternate $[\mathbf{A}^2]$ on i_1 and i_2. The result is symbolically

$$|\mathbf{a}_1\mathbf{b}_1|\,|\mathbf{a}_2\mathbf{b}_2|\mathbf{a}_3\mathbf{b}_3 \tag{47}$$

instead of $(\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)\mathbf{a}_3\mathbf{b}_3$ which we would have had by folding; and is a hexadic instead of a dyadic. Its components, however, are either zero or the quantities (7) with proper sign.

Example 7. If $q=n$ alternation produces the determinant of the components of the n vectors, multiplied by $\mathbf{E}$. Thus (47) is equal to $\mathbf{E}^2\mathbf{F}_1((\mathbf{A}^2))$.

Example 8. If $\mathbf{F}$ is a multiple covariant $\mathbf{EF}$ is a multiple covariant, provided the indices of $\mathbf{E}$ all belong to the same index of $\mathbf{A}$.

Some of the above examples can be at once extended to multiple concomitants in general, due regard being paid to the character of the indices. It appears to be true that all multiple concomitants of a single matrix $\mathbf{A}$ can be obtained from $\mathbf{E}^m[\mathbf{A}^q]$, where m and q are integers, and we employ folding, alternation, and isomerism.

20. Multiple concomitants and tensor calculus.

A multiply invariant number belonging to a p-way matrix $\mathbf{A}$, *e.g.*, any primary rank of $\mathbf{A}$ as defined above, or any primary rank of a multiple concomitant of $\mathbf{A}$, is evidently an invariant in the sense of tensor calculus when $\mathbf{A}$ is a tensor. Multiple concomitants are also related to tensor calculus by the following theorem:

Theorem XII. If $\mathbf{A}$ is a tensor and if $\mathbf{F}$ is a multiple concomitant of $\mathbf{A}$ then $\mathbf{F}$ is a *relative tensor* provided

1°. Any index of $\mathbf{F}$ cogredient with an index of $\mathbf{A}$ is written in the same manner (upper or lower) as the index of $\mathbf{A}$ to which it belongs.

2°. Any index of **F** contragredient to an index of **A** is written in the opposite way to the index of **A** to which it belongs.

Proof. With the notation of Art. 2 let μ_m be the value of μ when the transformation (1) is carried out on the index i_m. Let Δ_m be the determinant of the transformation on i_m. Let now p arbitrary non-singular transformations of the form (1) be carried out on the respective p indices of **A** simultaneously, the transformed **A** being called $'\mathbf{A}$. Let **F** be any multiple concomitant of **A** and let $'\overline{\mathbf{F}}$ be the polyadic whose components are the same functions of the components of $'\mathbf{A}$ that the components of **F** were of those of **A**. On the other hand let $'\mathbf{F}'$ be the transform of **F** under the p arbitrarily chosen transformations, with due regard to the cogredience or contragredience of the indices of **F** compared with their associated indices in **A** as defined in Art. 9. By successive application of (4) we have

$$'\overline{\mathbf{F}} = \Delta_1^{\mu_1}\Delta_2^{\mu_2}\cdots\Delta_p^{\mu_p}\,'\mathbf{F}'. \tag{48}$$

Since the p linear transformations are arbitrary, (48) will persist when these transformations are all the covariant transformation of tensor calculus, or when they are all the contravariant transformation, or when some are one, others the other. If **A** is a tensor and if i_m is a lower index write $\Delta_m = \Delta$ and $\mu_m = K_m$; but if i_m is an upper index write $\Delta_m = J = \Delta^{-1}$ and $\mu_m = -K_m$. Also let

$$K = K_1 + K_2 + \cdots + K_p.$$

Thus if **A** is a tensor (48) becomes

$$'\overline{\mathbf{F}} = \Delta^K\,'\mathbf{F}', \tag{49}$$

i.e., **F** satisfies the definition[8] of a relative tensor of weight K.

To take one of the simplest illustrations, if $n=2$ and $p=3$, and if $A_{i_1 i_2 i_3}$ is a covariant tensor, then $\mathbf{F}_1((\mathbf{A}^2))$ is a relative tensor $F_{j_1 j_2}$ of weight 2 with components given by (6). For by Art. 2 we have $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 0$. The three indices of **A** being lower we have $K = 2$.

Again, starting with the same triadic, if i_1 is an upper index with i_2 and i_3 lower, so that $A^{i_1}_{\cdot\, i_2 i_3}$ is a mixed tensor, then

[8] Veblen and Thomas, *Extensions of Relative Tensors*, Trans. Amer. Math. Soc., Vol. 26 (1924), p. 373.

$\mathbf{F}_1((\mathbf{A}^2))$ is a tensor; for $K_1=-1$, $K_2=+1$, and $K_3=0$, giving $K=0$.

As another example, if p is even and $A_{i_1}..i_p$ a covariant tensor, the full sign p-way determinant of the components is a relative invariant of weight p, for this determinant is a multiple invariant with $\mu=1$ on all indices (IV, p. 242). Denoting this determinant by $|A_{i_1}..i_p|$ we have

$$\mathbf{F}_1((\mathbf{A}^n))=\mathbf{E}\,|A_{i_1}..i_p| \tag{50}$$

by Theorem XIII of III, p. 236. This is a polyadic identity true no matter to what transformations the indices are subject, and $\mathbf{E}$ is the special polyadic of order n defined by (28). The left side, by Theorem V, is a multiple covariant of $\mathbf{A}$; since all its indices belong to the non-folded index i_p we have $\mu_p=0$ as in the illustrative example of Art. 2; and all the other μ's are $+1$ as in that example. Hence if $\mathbf{A}$ is a covariant tensor the left side is a relative tensor of weight $p-1$. Therefore by comparing the two sides $\mathbf{E}$ must in this case be a relative covariant tensor of weight -1.

This result may be proved directly. With the notation of Art. 2 we have $\bar{\mathbf{E}}=\mathbf{E}$ because the components of $\mathbf{E}$ are constant. On the other hand we have by definition

$$\mathbf{E}'=\begin{vmatrix} \mathbf{e}_a T_{1a} & \mathbf{e}_a T_{2a} & \cdots & \mathbf{e}_a T_{na} \\ \mathbf{e}_a T_{1a} & \mathbf{e}_a T_{2a} & \cdots & \mathbf{e}_a T_{na} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{e}_a T_{1a} & \mathbf{e}_a T_{2a} & \cdots & \mathbf{e}_a T_{na} \end{vmatrix}=\Delta\mathbf{E} \tag{51}$$

so that

$$\bar{\mathbf{E}}=\Delta^{-1}\mathbf{E}' \tag{52}$$

on the assumption that the n indices of $\mathbf{E}$ all belong to the same index of $\mathbf{A}$ and that the transformation $\mathbf{T}$ takes place on that index of $\mathbf{A}$; under these assumptions (52) holds no matter what non-singular linear transformation is denoted by $\mathbf{T}$. Hence if $\mathbf{T}$ is the covariant transformation of tensor calculus $\mathbf{E}$ is a relative covariant tensor of weight -1 as was to be shown. In this case $\mathbf{E}$ may properly be denoted by $E_{j_1}..j_n$.

Similarly if, in (50), we change the indices to upper, implying that $A^{i_1 \cdots i_p}$ is a contravariant tensor, (52) shows that **E** becomes a relative contravariant tensor of weight $+1$ or *tensor density* (because Δ^{-1} is now the Δ of Veblen and Thomas). In this case **E** may properly be denoted by $E^{j_1 \cdots j_n}$.

More generally, with the notation of Art. 12, $\mathbf{E}^{p-k}[\mathbf{A}^n]^{A \cdots G}$ is a tensor if **A** is a tensor provided each **E** is cogredient with one of the folded indices $A \cdots G$; for this quantity is a multiple covariant of **A** having $\mu = 0$ for any non-folded index: while $[\mathbf{A}^n]^{A \cdots G}$ has $\mu = 1$ for any folded index (proved as in Art. 2) and any **E** has $\mu = -1$ by (52), so that $\mu = 0$ also on any folded index of the given product. Hence $K = 0$. This result may also be proved by the use of the Murnaghan symbol $\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n}$ which vanishes unless the r's and s's are the same set of distinct integers and is ± 1 when the substitution to bring them to the same order is even or odd. It will be sufficient to take the case $p = 3$ and fold on i_2 and i_3. Note that

$$\delta^{r_1 \cdots r_n}_{s_1 \cdots s_n} = E^{r_1 \cdots r_n} E_{s_1 \cdots s_n} \tag{53}$$

where the two factors on the right are our contravariant **E** and covariant **E** respectively. Then

$$\delta^{\sigma_1 \cdots \sigma_n}_{s_1 \cdots s_n} \delta^{\tau_1 \cdots \tau_n}_{t_1 \cdots t_n} A_{r_1 \sigma_1 \tau_n} A_{r_2 \sigma_2 \tau_2} \cdots A_{r_n \sigma_n \tau_n} \tag{53_1}$$

$$= E^{\sigma_1 \cdots \sigma_n} E_{s_1 \cdots s_n} E^{\tau_1 \cdots \tau_n} E_{t_1 \cdots t_n} A_{r_1 \sigma_1 \tau_1} \cdots A_{r_n \sigma_n \tau_n} \tag{53_2}$$

$$= E_{s_1 \cdots s_n} E_{t_1 \cdots t_n} [\mathbf{A}^n]_r \tag{53_3}$$

$$= \mathbf{E}^2 [\mathbf{A}^n]_{i_1} \tag{53_4}$$

where one **E** belongs to i_2 and the other to i_3, and i_1 is not folded. Passing to tensor calculus, if the indices are transformed as indicated by their positions, **A** is a covariant tensor, δ is a tensor, hence (53_4) is a tensor as was to be proved.

It will be recalled that, when p is odd, $[\mathbf{A}^n]_{i_m}$ does *not* in general factor into **E** times a full sign determinant of the components $A_{i_1 \cdots i_p}$ (Cf. the example of Art. 2), but has for its non-vanishing

components a set of nth-order minors as shown in Art. 12, each with proper sign.

Remarks on the above examples. The fact that, if $A_{i_1 \cdots i_p}$ is a tensor with p even, the full sign determinant $|A_{i_1 \cdots i_p}|$ is a relative invariant is given for $p=2, n=4$, with all indices lower, by Eddington,[6] his interesting proofs being quite different from the above. Our $E^{j_1 \cdots j_n}$ is, for $n=4$, the same as his $\epsilon_{\alpha\beta\gamma\delta}$ which he also shows to be a tensor density, the above results being in entire agreement with his. They are also in agreement with a result but not with the phraseology of Dr. C. R. Cramlet[7] which in our notation is: if **A** is a tensor density **EA** is a tensor if **E** is the covariant (relative) tensor $E_{j_1 \cdots j_n}$. Since Dr. Cramlet states that he is in disagreement with Eddington it may be noted that Eddington's $\epsilon_{\alpha\beta\gamma\delta}$ satisfies Dr. Cramlet's equation (5.4) but not his (5.3). Also the J of Dr. Cramlet is the Δ of Veblen and Thomas, not Eddington's J. In another very powerful paper Dr. Cramlet employs expressions of the type $(53)_1$, which he calls a "determinant," expressly neglecting the distinction between what is usually called a determinant and a tensor whose non-vanishing components are determinants. It need hardly be remarked that the present investigation is based on precisely this distinction, the components of (53_1) comprising a set of minors of **A** as fully exemplified.

21. Illustrative examples.

Example 1. If the cubic rank of $A_{i_1 \cdots i_p}$ on a pair of indices is unity, the two-way rank on one of these indices is unity.

Proof. Note first the **lemma**: *if the general* (indeterminate) *product* (V, pp. 164, 165) *of two matrices* (polyadics) *vanishes, one of these matrices vanishes.* Suppose the cubic rank (Def. II) of $A_{i_1 \cdots i_p}$ on $i_1 i_2$ to be unity. By Theorem VII

$$[\mathbf{A}^2]^{i_1 i_2} = 0. \tag{54}$$

[6] *The Mathematical Theory of Relativity*, 2d Ed., Art. 48.

[7] Some General Determinant Theorems in Tensor Notation, Annals of Math., Vol. 27, No. 4 (Sept. 1926), p. 380, equation (5.6).

[8] Applications of the Determinant and Permanent Tensors to Determinants of General Class and Allied Tensor Functions, Amer. Jour. Math., Vol. XLIX, No. 1 (Jan. 1927), p. 89.

Expand **A** on its first two indices thus

$$\mathbf{A}=\mathbf{B}_{a_3a_4\cdots a_p}\mathbf{e}_{a_3}\mathbf{e}_{a_4}\cdots\mathbf{e}_{a_p} \tag{55}$$

where the **B**'s are n^{p-2} dyadics whose components are two-way blocks of the components of **A** with indices $i_3 \ldots i_p$ fixed in value throughout any one block (Art. 8). Following Gibbs and Naess the result of folding a pair of dyadics on both factors may be denoted by double cross. Substituting in (54) the expansion of **A** from (55) gives

$$\mathbf{B}_{a_3\cdots a_p}\overset{\times}{\times}\mathbf{B}_{\beta_3\cdots\beta_p}\mathbf{e}_{a_3}\mathbf{e}_{\beta_3}\mathbf{e}_{a_4}\mathbf{e}_{\beta_4}\cdots\mathbf{e}_{a_p}\mathbf{e}_{\beta_p}=0 \tag{56}$$

whence, because the $2p$-ads are independent,

$$\mathbf{B}_{a_3\cdots a_p}\overset{\times}{\times}\mathbf{B}_{\beta_3\cdots\beta_p}=0 \tag{57}$$

for all choices of the values of the indices. For each such choice the left side of (57) is a polyadic whose components are certain of the cubic minors of **A** on i_1i_2; *e.g.*, if $n=3$ the **B**'s are dyadics in the sense of Gibbs, the double cross product of two **B**'s is also a dyadic, and the components are a set of nine cubic determinants. Now take the cases when each α has the same value as the corresponding β, namely

$$\mathbf{B}_{i_3\cdots i_p}\overset{\times}{\times}\mathbf{B}_{i_3\cdots i_p}=0 \quad \text{(not summed)} \tag{58}$$

If the double cross product of a dyadic into itself vanishes the dyadic can be written as a dyad,[9] *i.e.*, all our **B**'s are dyads. Replace the $p-2$ non-folded indices by a $(p-2)$-partite index u running from *1* to n^{p-2}. Call the polyads $\mathbf{e}_{a_3}\cdots\mathbf{e}_{a_p}$ accordingly $\mathbf{E}_u$. Replace (55) by

$$\mathbf{A}=\mathbf{a}_1\mathbf{b}_1\mathbf{E}_1+\mathbf{a}_2\mathbf{b}_2\mathbf{E}_2+\mathbf{a}_3\mathbf{b}_3\mathbf{E}_3+\cdots+\mathbf{a}_N\mathbf{b}_N\mathbf{E}_N \tag{59}$$

where $N=n^{p-2}$. If the rank on i_1 exceeds unity there must be at least two non-vanishing terms on the right (V, p. 181, Theorem V). Suppose the first two terms to be actual terms, *i.e.*, assume $\mathbf{a}_1$,

[9] E. B. Wilson, On the Theory of Double Products and Strains in Hyperspace, Trans. Conn. Acad. Arts and Sci., Vol. 14 (Sept. 1908), p. 18.

$\mathbf{b}_1$, $\mathbf{a}_2$, and $\mathbf{b}_2$ not zero. Forming the equations (57) we have

$$\mathbf{a}_1 \times \mathbf{a}_2 \quad \mathbf{b}_1 \times \mathbf{b}_2 = 0 \tag{60a}$$

$$\mathbf{a}_1 \times \mathbf{a}_3 \quad \mathbf{b}_1 \times \mathbf{b}_3 = 0 \tag{60b}$$

$$\mathbf{a}_2 \times \mathbf{a}_3 \quad \mathbf{b}_2 \times \mathbf{b}_3 = 0 \tag{60c}$$

and so on. If the rank of $\mathbf{A}$ on i_1 exceeds unity at least one cross product of a pair of $\mathbf{a}$'s does not vanish (V, p. 170, Theorem I). Assume

$$\mathbf{a}_1 \times \mathbf{a}_2 \neq 0 \tag{61}$$

whence by (60*a*) and the above lemma we must have

$$\mathbf{b}_1 \times \mathbf{b}_2 = 0, \tag{62}$$

i.e., $\mathbf{b}_1$ and $\mathbf{b}_2$ are parallel since neither is zero. Thus (60*b*) and (60*c*) may be written

$$\mathbf{a}_1 \times \mathbf{a}_3 \quad \mathbf{b}_1 \times \mathbf{b}_3 = 0 \tag{60b}$$

$$\mathbf{a}_2 \times \mathbf{a}_3 \quad \mathbf{b}_1 \times \mathbf{b}_3 = 0 \tag{63}$$

but if the third term of (59) actually exists, *i.e.*, if $\mathbf{a}_3$ and $\mathbf{b}_3$ are both different from zero, $\mathbf{a}_3$ is not parallel to both the non-parallel vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ so that we must have

$$\mathbf{b}_1 \times \mathbf{b}_3 = 0. \tag{64}$$

It has thus been shown that if $\mathbf{a}_1 \times \mathbf{a}_2$ is not zero and if the first three terms of (59) exist the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are parallel. Similarly, treating any other actual term as the third has just been treated, all $\mathbf{b}$'s are parallel. That is, $\mathbf{A}$ is of rank unity on i_2 if not on i_1. Similarly if the cubic rank of $\mathbf{A}$ on *any* two indices is unity the rank on one of these indices is unity.

Example 2. The above example extends, by parallel reasoning, to 3-way rank on a pair of multipartite indices (Def. III). The result may be stated as

Theorem XIII. If the 3-way multiplex rank of a matrix $A_{i_1 \cdots i_p}$ is unity on a pair of multipartite indices, the two-way multiplex rank on one of these indices is unity.

In extending the proof a cross will denote multiplex folding instead of simple folding (Art. 16). Two polyadics are "parallel" when their cross product vanishes, *i.e.*, their multiplex space complement is zero, and they are linearly related. If a double polyadic $B_{u_1 u_2}$ (multipartite indices) multiplied by double cross (in the sense just mentioned) into itself gives zero it can be written

$B_{u_1}B_{u_2}$, Wilson's proof (loc. cit.) applying without formal change. The rest of the proof is unaltered except that instead of vectors $\mathbf{a}_1$, $\mathbf{b}_1$, etc., we have polyadics of order t_1 and t_2; all indices not folded remain simple from $i_{t_1+t_2+1}$ to i_p, instead of i_3 to i_p.

Example 3. If the cubic rank of $\mathbf{A}$ on a pair of indices is *2*, the two-way ranks on the separate indices may both exceed *2*. For instance the triadic

$$(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_1+(\mathbf{e}_2\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_2 \tag{65}$$

has cubic rank *2* on the first two factors but two-way rank *3* on each of these factors separately.

Example 4. Let there be $\sigma-1$ multipartite (or simple) indices and let the σ-way rank on these indices be R_σ. On any $\tau-1$ of these indices $(1<\tau<\sigma)$ we have a τ-way rank which may be called *subordinate* to R_σ.

Theorem XIV. R_σ cannot exceed the smallest of its subordinate ranks.

For let the $\sigma-1$ indices be A, B, $\cdots$ G, and suppose the smallest of the subordinate ranks to be rank q on indices A $\cdots$ D. Then by Theorem VII

$$[\mathbf{A}^{q+1}]^{\text{A}\cdots\text{D}}=0 \tag{66}$$

and therefore

$$[\mathbf{A}^{q+1}]^{\text{A}\cdots\text{G}}=0 \tag{67}$$

which proves the theorem.

Example 5. If R_3 (A, B) is the cubic rank of a non-vanishing matrix $A_{i_1}\ldots i_p$ on a pair of indices (simple or multipartite) A and B, and if R_2(A) and R_2(B) are the two-way ranks on the separate indices, it is an immediate consequence of Theorems XIII and XIV that the equations

$$R_3(\text{A, B})-1=0 \tag{68a}$$

and

$$[R_2(\text{A})-1][R_2(\text{B})-1]=0 \tag{68b}$$

are equivalent. To take the simplest case, if the three quantities (7), which are the cubic 2d-order minors of the indices i_1i_2 of $A_{i_1i_2i_3}$,

are all zero, the six two-way minors on i_1 are all zero, or else the six on i_2 are all zero; and conversely. This may be verified by ordinary algebra.

Example 6. As another corollary to Theorem XIII, if the cubic rank of a pair of indices i_a and i_b is unity we may write one or the other of the two developments.

$$\mathbf{A} = \mathrm{A}_{i_1 \cdots i_p} = \mathrm{a}_{i_a} \mathrm{B}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_p} \tag{69a}$$

$$\mathrm{A}_{i_1 \cdots i_p} = \mathrm{b}_{i_b} \mathrm{B}_{i_1 \cdots i_{b-1} i_{b+1} \cdots i_p} \tag{69b}$$

where a_{i_a} and b_{i_b} are vectors. A similar result holds with multipartite indices.

Example 7. Writing $(\mathbf{ab})$ for $\mathrm{a}_1\mathrm{b}_2 - \mathrm{a}_2\mathrm{b}_1$, etc., we have identically

$$\begin{vmatrix} \mathrm{a}_1\mathrm{u}_1 & \mathrm{a}_1\mathrm{u}_2 & \mathrm{a}_2\mathrm{u}_1 & \mathrm{a}_2\mathrm{u}_2 \\ \mathrm{b}_1\mathrm{v}_1 & \mathrm{b}_1\mathrm{v}_2 & \mathrm{b}_2\mathrm{v}_1 & \mathrm{b}_2\mathrm{v}_2 \\ \mathrm{c}_1\mathrm{x}_1 & \mathrm{c}_1\mathrm{x}_2 & \mathrm{c}_2\mathrm{x}_1 & \mathrm{c}_2\mathrm{x}_2 \\ \mathrm{d}_1\mathrm{y}_1 & \mathrm{d}_1\mathrm{y}_2 & \mathrm{d}_2\mathrm{y}_1 & \mathrm{d}_2\mathrm{y}_2 \end{vmatrix} \tag{70a}$$

$$= \frac{1}{3} \Big\{ [(\mathbf{ac})(\mathbf{bd}) + (\mathbf{ad})(\mathbf{bc})](\mathbf{uv})(\mathbf{xy}) \tag{70b}$$

$$- [(\mathbf{ab})(\mathbf{cd}) + (\mathbf{ad})(\mathbf{cb})](\mathbf{ux})(\mathbf{vy}) \tag{70c}$$

$$+ [(\mathbf{ab})(\mathbf{dc}) + (\mathbf{ac})(\mathbf{db})](\mathbf{uy})(\mathbf{vx}) \Big\} \tag{70d}$$

$$= \frac{1}{3} \Big\{ -(\mathbf{ab})(\mathbf{cd})[(\mathbf{ux})(\mathbf{vy}) + (\mathbf{uy})(\mathbf{vx})] \tag{70e}$$

$$+ (\mathbf{ac})(\mathbf{bd})[(\mathbf{uv})(\mathbf{xy}) + (\mathbf{uy})(\mathbf{xv})] \tag{70f}$$

$$- (\mathbf{ad})(\mathbf{bc})[(\mathbf{ux})(\mathbf{yv}) + (\mathbf{uv})(\mathbf{yx})] \Big\} \tag{70g}$$

This identity, since (70*a*) is of the same form as (38), verifies the statement made in Art. 16 in regard to expansion of this determinant.

Proof. For brevity denote (70a) by D and write

$$\Delta = T_{11}T_{22} - T_{12}T_{21}, \quad \Omega = \frac{\partial^2}{\partial T_{11}\partial T_{22}} - \frac{\partial^2}{\partial T_{12}\partial T_{21}} \tag{71}$$

and operate by Ω^2 on both sides of the identity

$$\begin{vmatrix} T_{a1}a_a u_1 & T_{a1}a_a u_2 & T_{a2}a_a u_1 & T_{a2}a_a u_2 \\ T_{a1}b_a v_1 & T_{a1}b_a v_2 & T_{a2}b_a v_1 & T_{a2}b_a v_2 \\ T_{a1}c_a x_1 & T_{a1}c_a x_2 & T_{a2}c_a x_1 & T_{a2}c_a x_2 \\ T_{a1}d_a y_1 & T_{a1}d_a y_2 & T_{a2}d_a y_1 & T_{a2}d_a y_2 \end{vmatrix} = \Delta^2 D \tag{72}$$

Operating on the right side gives 12D as is well known.[10] To operate on the left side make first a Laplacian development by minors of the first two columns, *i.e.*, take the sum of six determinants of which the first is

$$\begin{vmatrix} T_{a1}a_a u_1 & T_{a1}a_a u_2 & 0 & 0 \\ T_{a1}b_a v_1 & T_{a1}b_a v_2 & 0 & 0 \\ 0 & 0 & T_{a2}c_a x_1 & T_{a2}c_a x_2 \\ 0 & 0 & T_{a2}d_a y_1 & T_{a2}d_a y_2 \end{vmatrix} \tag{73}$$

Abbreviating $\dfrac{\partial}{\partial T_{11}}$ by ∂_{11}, etc., we have

$$\partial_{11}T_{a1}a_a = a_1 \quad \partial_{21}T_{a1}a_a = a_2 \quad \partial_{12}T_{a1}a_a = 0, \text{ etc.}$$

whence operating on (73) with $\partial_{11}\partial_{11}$ gives twice the determinant

$$\begin{vmatrix} a_1u_1 & a_1u_2 & 0 & 0 \\ b_1v_1 & b_1v_2 & 0 & 0 \\ 0 & 0 & T_{a2}c_a x_1 & T_{a2}c_a x_2 \\ 0 & 0 & T_{a2}d_a y_1 & T_{a2}d_a y_2 \end{vmatrix} \tag{74}$$

[10] Grace and Young, Algebra of Invariants, Art. 26.

and operating on (74) with $\partial_{22}\partial_{22}$ gives four times the determinant

$$\begin{vmatrix} a_1u_1 & a_1u_2 & 0 & 0 \\ b_1v_1 & b_1v_2 & 0 & 0 \\ 0 & 0 & c_2x_1 & c_2x_2 \\ 0 & 0 & d_2y_1 & d_2y_2 \end{vmatrix} \tag{75}$$

that is $4a_1b_1c_2d_2$ **(uv) (xy)**. Similarly $\partial_{12}\partial_{12}\partial_{21}\partial_{21}$ yields $4a_2b_2c_1d_1$ **(uv)(xy)**. Finally $\partial_{12}\partial_{21}\partial_{22}\partial_{11}$ yields

$$[a_1b_2c_1d_2+a_1b_2c_2d_1+a_2b_1c_1d_2+a_2b_1c_2d_1]\mathbf{(uv)(xy)}$$

Collecting results we find the yield of Ω^2 on (73) to be

$$2[\mathbf{(ac)(bd)}+\mathbf{(ad)(bc)}]\mathbf{(uv)(xy)}.$$

Continuing the Laplacean development the required identity is obtained.

Example 8. In Example 7, instead of a series of letters a, b, c, d, we may stipulate development in order of rows and write $\mathbf{e}_i=a_i=b_i=c_i=d_i$. Similarly let $\mathbf{f}_i=u_i=v_i=x_i=y_i$, with the same requirement. Then if $\mathbf{e_1e_2}-\mathbf{e_2e_1}=\mathbf{E}$ and $\mathbf{f_1f_2}-\mathbf{f_2f_1}=\mathbf{F}$ the product **(ab)(cd)** becomes the direct product **EE** or $\mathbf{E^2}$. The product **(ac)(bd)** becomes the multiple indeterminate product $[\mathbf{E^2}]$, that is

$$[\mathbf{E^2}]=\mathbf{e_1e_1e_2e_2}-\mathbf{e_1e_2e_2e_1}-\mathbf{e_2e_1e_1e_2}+\mathbf{e_2e_2e_1e_1} \tag{76}$$

The product **(ad)(bc)**, by virtue of the identity

$$\mathbf{(ad)(bc)}=\mathbf{(ac)(bd)}-\mathbf{(ab)(cd)} \tag{77}$$

becomes $[\mathbf{E^2}]-\mathbf{E^2}$. In a similar manner the products **(uv)(xy)**, **(ux)(vy)**, and **(uy)(vx)** become $\mathbf{F^2}$, $[\mathbf{F^2}]$, and $[\mathbf{F^2}]-\mathbf{F^2}$. Collecting results the identity (70) becomes

$$\begin{vmatrix} \mathbf{e_1f_1} & \mathbf{e_1f_2} & \mathbf{e_2f_1} & \mathbf{e_2f_2} \\ \mathbf{e_1f_1} & \mathbf{e_1f_2} & \mathbf{e_2f_1} & \mathbf{e_2f_2} \\ \mathbf{e_1f_1} & \mathbf{e_1f_2} & \mathbf{e_2f_1} & \mathbf{e_2f_2} \\ \mathbf{e_1f_1} & \mathbf{e_1f_2} & \mathbf{e_2f_1} & \mathbf{e_2f_2} \end{vmatrix} = [\mathbf{E^2}]\mathbf{F^2}-\mathbf{E^2}[\mathbf{F^2}] \tag{78}$$

Example 9. As an obvious corollary, the determinant (70*a*) is identically equal to

$$(\mathbf{ac})(\mathbf{bd})(\mathbf{uv})(\mathbf{xy})-(\mathbf{ab})(\mathbf{cd})(\mathbf{ux})(\mathbf{vy}). \tag{70h}$$

Example 10. The duplex space complement of a binary dyad **dy** is by definition

$$\begin{vmatrix} \mathbf{e}_1\mathbf{f}_1 & \mathbf{e}_1\mathbf{f}_2 & \mathbf{e}_2\mathbf{f}_1 & \mathbf{e}_2\mathbf{f}_2 \\ \mathbf{e}_1\mathbf{f}_1 & \mathbf{e}_1\mathbf{f}_2 & \mathbf{e}_2\mathbf{f}_1 & \mathbf{e}_2\mathbf{f}_2 \\ \mathbf{e}_1\mathbf{f}_1 & \mathbf{e}_1\mathbf{f}_2 & \mathbf{e}_2\mathbf{f}_1 & \mathbf{e}_2\mathbf{f}_2 \\ \mathrm{d}_1\mathrm{y}_1 & \mathrm{d}_1\mathrm{y}_2 & \mathrm{d}_2\mathrm{y}_1 & \mathrm{d}_2\mathrm{y}_2 \end{vmatrix} \tag{79}$$

where the basis vectors of the second index are written $\mathbf{f}_1$ and $\mathbf{f}_2$ instead of $\mathbf{e}_1$ and $\mathbf{e}_2$ merely to assist the eye. Using the method of Example 8, write $\mathbf{e}_1=\mathrm{a}_1=\mathrm{b}_1=\mathrm{c}_1$, $\mathbf{e}_2=\mathrm{a}_2=\mathrm{b}_2=\mathrm{c}_2$, $\mathbf{f}_1=\mathrm{u}_1=\mathrm{v}_1=\mathrm{x}_1$, and $\mathbf{f}_2=\mathrm{u}_2=\mathrm{v}_2=\mathrm{x}_2$ in (70a) and (70h), and require m.i.p. development by rows. The determinant (**bd**) becomes $\mathbf{e}_1\mathrm{d}_2-\mathbf{e}_2\mathrm{d}_1$ or (**d**), the simple space complement of **d**. Similarly (**xy**) becomes (**y**), the simple space complement of **y**; and (**cd**) and (**vy**) become (**d**) and (**y**). The product (**ac**)(**bd**) becomes the m.i.p. [**E**(**d**)], that is, the multiple indeterminate product of **E** into (**d**), but since d_1 and d_2 are scalars this merely means that we interchange the order of the last two vector factors in each term of **E**(**d**). Hence

$$[\mathbf{E}(\mathbf{d})]=\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathrm{d}_2-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathrm{d}_1-\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathrm{d}_2+\mathbf{e}_2\mathbf{e}_2\mathbf{e}_1\mathrm{d}_1 \tag{80}$$

and similarly for the product (**ux**)(**vy**). Thus (79) takes the form

$$[\mathbf{E}(\mathbf{d})]\mathbf{F}(\mathbf{y})-\mathbf{E}(\mathbf{d})[\mathbf{F}(\mathbf{y})] \tag{79a}$$

Example 11. Similarly the duplex space complement of two binary dyads **cx** and **dy** may be developed as

$$(\mathbf{c})(\mathbf{d})\mathbf{F}(\mathbf{xy})-\mathbf{E}(\mathbf{cd})(\mathbf{x})(\mathbf{y}) \tag{81}$$

Example 12. Similarly the duplex space complement of three binary dyads **bv**, **cx**, and **dy** may be developed as

$$(\mathbf{c})(\mathbf{bd})(\mathbf{v})(\mathbf{xy})-(\mathbf{b})(\mathbf{cd})(\mathbf{x})(\mathbf{vy}) \tag{82}$$

It may be remarked that Examples 10–12 illustrate the statement of Art. 16 that multiplex space complements can be expressed in terms of simple space complements and the determinant **E**. They are also in agreement with the surmise at the close of Art. 19 that multiple concomitants are in general formed from $\mathbf{E}^m[\mathbf{A}^q]$.

Example 13. The left member of (78), which is an 8-adic, may be denoted either by $E_{u_1u_2u_3u_4}$ or by $E_{j_1j_5,j_2j_6,j_3j_7,j_4j_8}$, where u_1, u_2, etc., are bipartite indices corresponding to the pairs of simple indices j_1j_5, j_2j_6, etc., and run from *1* to *4*. We might define this 8-adic without reference to anything but its matrix by saying that any component is zero when a pair of bipartite indices are alike in value and is ± 1 when the permutation needed to bring the bipartite indices to natural order is even or odd.

Example 14. With the notation of Example 13, the identity (78) may be written

$$E_{j_1j_5,j_2j_6,j_3j_7,j_4j_8} = E_{j_1j_2}E_{j_3j_4}E_{j_5j_6}E_{j_7j_8} - E_{j_1j_2}E_{j_3j_4}E_{j_5j_7}E_{j_6j_8} \tag{83}$$

Example 15. With the same notation the duplex space complement (33) of a pair of binary dyadics may be written

$$E_{j_1j_5,j_2j_6,\alpha_2\alpha_7,\alpha_4\alpha_8}\,a_{\alpha_2\alpha_7}b_{\alpha_4\alpha_8} \tag{84}$$

summed on the Greek indices.

Example 16. With the same notation we may write, instead of the symbolic form (43), the explicit form

$$E_{\sigma\tau}E_{j_1j_5,j_2j_6,\alpha_2\alpha_7,\alpha_4\alpha_8}A_{\sigma\alpha_2\alpha_7i_4}A_{\tau\alpha_4\alpha_8i_4'}. \tag{85}$$

Example 17. Similar to Example 13, the multiplex **E** corresponding to binary triadics, *i.e.*, the determinant of the 8th order whose identical rows are the triads $\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1, \cdots, \mathbf{e}_2\mathbf{e}_2\mathbf{e}_2$ in natural order, is a 24-adic whose 2^{24} components may be denoted by $E_{j_1j_9j_{17},j_2j_{10}j_{18}\cdots j_8j_{16}j_{24}}$. It might also be written $E_{i_1j_1k_1\cdots i_8j_8k_8}$ if we understand that all i's precede all j's, with k's last. Again it may be written $E_{u_1\cdots u_8}$ the indices being tripartite, and running from *1* to *8*. In this latter case its definition is verbally the same as if it were the simple $E_{j_1\cdots j_8}$ of an n-space (Art. 14).

Example 18. The multiplex **E** for ternary dyadics is an 18-adic $E_{j_1 j_{10} \cdots j_9 j_{18}}$.

Example 19. The multiplex **E** for tetradics in 4-space is a determinant whose identical rows are the tetrads $\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 \cdots \mathbf{e}_4\mathbf{e}_4\mathbf{e}_4\mathbf{e}_4$ in natural order. Thus it is a 64-adic whose 2^{64} components may be denoted by

$$E_{h_1 i_1 j_1 k_1 \cdots h_{16} i_{16} j_{16} k_{16}} \tag{86}$$

with the understanding that the 64 indices are actually to be in four groups alphabetically arranged. Any set of four, beginning with $h_1 i_1 j_1 k_1$, may be taken as a 4-partite index giving

$$E_{u_1 \cdots u_{16}} \tag{87}$$

the indices now running from *1* to *16*. In this form the definition is verbally the same as that of a simple $E_{j_1 \cdots j_{16}}$ in 16-space.

Example 20. For any values of n and p we may let $N = n^p$; hence the multiplex **E** is by the definition (32) a determinant of order N, and is a pN-adic. To express the matrix of n^{pN} components each index may carry two subscripts separated by a comma, thus

$$E_{j_{1,1} j_{2,1} \cdots j_{p,1} \cdots \cdots j_{1,N} j_{2,N} \cdots j_{p,N}} \tag{88}$$

if we understand that the order of indices is actually the natural order of the bipartite subscripts. If each group of p indices as written be replaced by a p-partite index we have

$$E_{u_1 \cdots u_N} \tag{89}$$

so that instead of defining this pN-adic by the determinant (32) it may be defined to have any component equal to zero if any two multipartite indices are alike in value and any component equal to $+1$ or to -1 according as the permutation to bring the N multipartite indices into natural order is even or is odd.

Example 21. With the notation of Example 20, let $\sigma_1 \cdots \sigma_N$ correspond to $u_1 \cdots u_N$. The multiplex space complement of q p-adics may then be written

$$E_{u_1 \cdots u_{N-q} \sigma_{N-q+1} \cdots \sigma_N} A_{\sigma_{N-q+1}} \cdots G_{\sigma_N} \tag{90}$$

whence any expression obtained by multiplex folding may be written in explicit form in a manner similar to (85).

PARALLEL GEODESICS

By C. L. E. Moore

1. **Two dimensions.** Let us first examine what relations the coefficients of the element of arc

$$ds^2 = a_{rs} du^r du^s \quad r, s = 1, 2 \tag{1}$$

must satisfy in order that the coördinate curves may be parallel in the sense that, if a tangent to one parameter line is moved by Levi-Civita parallelism along the parameter curve of the other system it will be tangent to a parameter curve in its new position. Thus we move the vector $(0, t^2)$ along the direction $(du^1, 0)$. The components of the change in the vector caused by the displacement are

$$\delta t^i = dt^i + \left\{ \begin{matrix} 1 & 2 \\ i & \end{matrix} \right\} t^2 \, du^1 \tag{2}$$

and if the vector is to remain tangent to the curve whose parameter is u^2, it is necessary for the component δt^1 to vanish. Since $dt^1 = 0$, the necessary and sufficient condition for parallelism in this sense is

$$\left\{ \begin{matrix} 1 & 2 \\ 1 & \end{matrix} \right\} = 0. \tag{3}$$

Likewise the necessary and sufficient condition that the curves of parameter u^1 are parallel is

$$\left\{ \begin{matrix} 1 & 2 \\ 2 & \end{matrix} \right\} = 0. \tag{4}$$

If both these conditions, (3), (4) are satisfied the coördinates form a Tchebychef net. Hence *the lines of a Tchebychef net are parallel.*

In this case it is easy to see that the element of arc takes the simple form

$$ds^2 = (du^1)^2 + 2F du^1 du^2 + (du^2)^2$$

and we see that the surface is divided into figures with the four sides equal and the opposite sides parallel (in the above sense).

Similarly, if

$$\left\{ \begin{matrix} 1 & 1 \\ 2 \end{matrix} \right\} = 0, \tag{5}$$

the curves whose parameter is u_1 are geodesics. If both (4) and (5) are satisfied this set of parameter curves are parallel geodesics. But in this case the surface must be one of zero curvature.[1] Hence, *developable surfaces are the only ones which admit a set of parallel geodesics.*

From this theorem it is obvious that *on a non-developable surface there is no set of directions along which the tangents to an arbitrary geodesic can be moved by parallelism, so as to envelop a new geodesic.* For if there were such a set of directions we could then determine a set of parallel geodesics.

If (3) is satisfied and $a_{12}=0$, it is easily seen that the element of arc can be reduced to

$$(du^1)^2+a_{22}(du^2)^2$$

and hence the curves of parameter u^1 are geodesics. Hence if a set of parameter lines are parallel by moving in directions perpendicular to themselves they must be the orthogonal trajectories of a set of geodesics.

2. ***N*-dimensions.** We will consider only the case of geodesics. That is, given a congruence of geodesics which is left invariant by some parallel displacement, we will take this congruence as one set of parameter curves, say x^1, and let the lines along which the parallel displacement takes place be another set of parameter curves, say x^2. Now if we write the element of arc in the form

$$ds^2=a_{rs}dx^r dx^s \tag{6}$$

the condition that the curves whose parameter is x^1 are geodesics is

$$\left\{ \begin{matrix} 1 & 1 \\ m \end{matrix} \right\} = 0,\ m=2,\ 3,\ \cdots n. \tag{7}$$

If a curve of the congruence x^1 is moved by parallelism into another curve of the congruence

$$\left\{ \begin{matrix} 1 & 2 \\ m \end{matrix} \right\} = 0,\ m=2,\ 3,\ \cdots n. \tag{8}$$

[1] Bianchi, Geometria Differenziale, Vol. I, p. 77 (1903).

From (7) and (8) we see that, if a geodesic congruence is left invariant by one infinitesimal parallel displacement it is left invariant by an infinite number. For if we displace dx^1 by parallelism in the direction $\lambda dx^1+\mu dx^2$, the component of the displacement in the direction x^m $(m\neq 1)$ is

$$\lambda \left\{ \begin{matrix} 1 & 1 \\ m & \end{matrix} \right\} (dx^1)^2+\mu \left\{ \begin{matrix} 1 & 2 \\ m & \end{matrix} \right\} dx^1\, dx^2$$

which vanishes for all values of λ and μ. Now in the x^1 x^2-coördinate surfaces the x^1 curves being geodesics of the enveloping space are geodesics of the surface, but the only surfaces admitting a set of parallel geodesics are those of curvature zero. Hence if a geodesic congruence admits a parallel displacement the variety V_n contains an $(n-2)$-parametered set of 2-surfaces of zero curvature, each containing a pencil of parallel geodesics of V_n. The x^2 parameter lines can be chosen as any lines lying on the above surfaces and we shall therefore choose them orthogonal to the x^1 parameter lines, and we can also choose the parameter x^1 as arc length. Then the following equations are satisfied

$$\left\{ \begin{matrix} 1 & 1 \\ 1 & \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 2 \\ 1 & \end{matrix} \right\} = 0,$$

from which we easily get

$$\left\{ \begin{matrix} 1 & 1 \\ m & \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 2 \\ m & \end{matrix} \right\} = 0,\ m=1, \cdots n. \tag{9}$$

From the first of these equations

$$\frac{\partial a_{1m}}{\partial x^1} = 0. \quad m=1, \cdots n, \tag{10}$$

and from the second, since $a_{12}=0$,

$$\frac{\partial a_{1m}}{\partial x^2} = \frac{\partial a_{2m}}{\partial x^1} = 0,\ m=1, \cdots n. \tag{11}$$

If the congruence of geodesics is normal, that is if

$$a_{1m}=0,\ m=2, 3, \cdots$$

we see that the coefficients a_{2m} are not functions of x^1 and hence the element of arc can be written

$$ds^2 = (dx^1)^2 + \sum_{2}^{n} a_{rs}\, dx^r\, dx^s,$$

where the coefficients a_{2m} are independent of x^1.

If the normal congruence of curves of parameter x^1 has two parallels and the curves of the second displacement have x^3 for parameter, then by the above argument the coefficients a_{3m} are independent of x^1 and the V_n is generated in two ways by 2-surfaces of zero curvature having a generation of parallel geodesics of V_n. Also since a_{22} a_{23}, a_{33} are independent of x^1 we see that the space $x_4 = c_4, \cdots x_n = c_n$ is a space containing a generation of totally geodesic surfaces.[2] More generally if the congruence can be moved in k directions into parallel geodesics a_{rs} $(r = 1, 2, \cdots k,\ s = 1 \cdots n)$ are independent of x^1 and the space $x_{k+1} = c_{k+1}, \cdots x_n = c_n$ contains one generation of totally geodesic surfaces of that space.

The coördinate surfaces x^1 x^2 which contain a set of parallel geodesics play the role of cylinders in euclidean space. All the directions tangent to such a surface are moved by parallelism into a linear pencil of directions and hence are determined by the position of two of them. That is if we wish to move the tangent plane by parallelism along the geodesic we need only move the element dx^1 and dx^2, and from equations (7) and (8) we see that they move into tangent directions and hence these surfaces have a second property of cylinders, *viz.*, tangent planes at two points of the same generator are parallel.

However, these are not the only surfaces containing a set of geodesics with parallel tangent planes, for this does not require $\left\{ \begin{matrix} 1 & 2 \\ & 2 \end{matrix} \right\}$ to vanish.

If a congruence of geodesics can be moved by parallelism along $(n-1)$ congruences into itself, by the argument used before we see that it is moved into itself by any paths whatsoever and consequently this becomes the case considered by Eisenhart.[3]

[2] See Eisenhart, Riemannian Geometry, p. 71.
[3] Riemannian Geometry, p. 67.

In this case the congruence must be normal and the element of arc can be put into the form

$$ds^2 = (dx^1)^2 + \sum_2^n a_{rs}\, dx^r\, dx^s ,$$

where the coefficients a_{rs} are independent of x^1. The hypersurfaces $x^1 = \text{const.}$ are totally geodesic. Any one parameter family of geodesics of the congruence is a surface of zero curvature and the geodesics of the congruence which cuts a geodesic of one of the totally geodesic hypersurfaces will generate a totally geodesic surface of zero curvature. This is evident, since any line tangent to such a surface will be moved by parallelism (with respect to the space V_n) into a tangent line. From this we see that *the surface formed by all the lines of the congruence, x^1, of parallel geodesics, which cut an arbitrary geodesic, will be a totally geodesic surface of zero curvature.* For if we take a point P of the arbitrary geodesic and the totally geodesic hypersurface through P, then one of the orientations determined by the geodesic of the parallel congruence passing through P and a geodesic of the hypersurface will contain the arbitrary geodesic. Then we have a set of totally geodesic surfaces of zero curvature which will contain all the geodesics of the V_n, and the geodesics must then cut all the geodesics of the parallel congruence (which they intersect) at a constant angle and are then moved by parallelism along the lines of the parallel congruence into geodesics of V_n. Hence, *each geodesic of V_n has a pencil of geodesics parallel to it.*

A QUALITATIVE DEFINITION OF THE SUB- AND SUPER-HARMONIC FUNCTIONS[1]

By PHILIP FRANKLIN

1. **Introduction.** An integral characterization of functions satisfying Poisson's equation

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = -4\pi\rho, \ \rho \leq 0 \text{ (or } \rho \geq 0), \tag{1}$$

the density ρ as indicated never changing sign, has recently been given by F. Riesz,[2] which, essentially, identifies them with the sub- (or super-) harmonic functions previously studied in other connections. These functions are defined as follows. A function $u(x_1, \cdots x_n)$ defined for the interior of a domain D, and upper semi-continuous (*i.e.* the limit of a decreasing series of continuous functions) never becoming infinite too rapidly (faster than $1/r^{n-2}$ or $\log r$ for $n=2$, r being the distance from the point in question) is called *sub-harmonic* if, for every closed domain D' entirely inside of D and every function $U(x_1, \cdots x_n)$ harmonic in D' and such that

$$u(x_1, \cdots x_n) \leq U(x_1, \cdots x_n) \tag{2}$$

on the boundary of D', the same relation holds for the interior of D'. The negative of a sub-harmonic function is a super-harmonic function. To express the relation with (1) more precisely, every sub- (super-) harmonic function is, to within an additive harmonic function the potential function corresponding to a distribution of negative (positive) mass whose density is an integrable function. Conversely, every such potential function is a sub- (super-) harmonic function.

As the sub- and super-harmonic functions are thus precisely those which appear in the potential problems of a theory, such as the Newtonian theory of gravitation, where the density is of fixed sign, it is of interest to define them by a simple set of properties. The object of this paper is to formulate a set of postulates which

[1] Presented to the American Mathematical Society, May 7, 1927.

[2] F. Riesz, Acta Mathematica, vol. 48 (1926), p. 329 f.

are qualitative in the sense of involving neither derivatives nor integrals, and completely characterize the functions in question. The postulates are analogous to a similar set previously given by the writer[3] for the potential functions.

2. **Continuous sub- and super-harmonic functions.** As the postulates are much simpler for the case of continuous functions than for the general case, we first state them for this special class. Consider then, a class of functions of n variables, x_i, and let each function have associated with it an n-dimensional domain D. We assume

(1) Each function is continuous in the n variables at all interior points of its domain D.

(2) If D_1 and D_2, the domains for two functions of the class $u_1(x_i)$ and $u_2(x_i)$, have a domain D_3 in common, then any linear combination of these functions with positive coefficients, $A^2u_1+B^2u_2$, is a function of the class, whose region contains all the points of D_3.

(3) If an orthogonal transformation of the variables converts the function $u(x_i)$ with domain D into $u'(x_i')$, this latter function is a member of the class, its domain being the domain D expressed in the new variables.

(4) The functions $u(x_i)=1$, and $u(x_i)=-1$ are functions of the class, their domains being the whole of n-space.

(5) If a given (n-dimensional) sphere lies entirely inside each domain D_j, belonging to $u_j(x_i)$, these u_j constituting an infinite sequence of functions of the class, and if, for points on the surface of the sphere, x_{si}, $\lim_{j=\infty} u_j(x_{si})=0$, the limit being approached uniformly over the surface, then the sequence of values of the functions $u_j(x_{oi})$ at the center of the sphere x_{oi} cannot approach a *positive* limit.

We shall show that any family of functions satisfying these five postulates is necessarily a class of sub-harmonic functions. If we replace (5) by (5'), identical with (5) except for the substitution of "*negative* limit" for the last two words, a set is obtained which characterizes the super-harmonic functions. The set we previously gave for the harmonic functions was practically this set,

[3] Bulletin of the American Math. Soc., vol. 30 (1924), pp. 41–50.

with (5) and (5′) both holding. This is in conformity with the equivalence of functions which are simultaneously sub- and super-harmonic with harmonic functions.

By the argument we gave[4] for the harmonic case, we may show that, if $u(x_i)$ satisfies the first four postulates, and S is any sphere entirely within its domain D, with center at x_{oi}, there exists a sequence of functions of the class, $u_j(x_i)$ such that

$$|u_j(x_{oi}) - \text{average } u \text{ over } S| < \epsilon_j, \quad \lim_{j=\infty} \epsilon_j = 0, \tag{3}$$

and

$$u_j(x_{oi}) = u(x_{oi}). \tag{4}$$

As the sequence of functions $G_j(x_i) = u_j(x_i) - \text{av } u$ over S are all members of the class, by postulates (2) and (4), and satisfy the other conditions of postulate (5), we must have

$$u(x_{oi}) - \text{av } u \text{ over } S = G_j(x_{oi}) = \lim_{j=\infty} G_j(x_{oi}) \leq 0. \tag{5}$$

But, as F. Riesz has shown[5] (cf. the argument following (18) below) that every continuous function having its value at the center of every sphere entirely in its domain less than or equal to its average over the surface of the sphere is necessarily sub-harmonic, it follows from the inequality (5) that $u(x_i)$ is sub-harmonic.

We note that, as any continuous sub-harmonic function obviously satisfies the postulates (1) to (5), the widest class of functions satisfying them is the totality of continuous sub-harmonic functions. Just as for the harmonic functions, we may replace the last two postulates by a single one, which, though more stringent, is simpler. Namely

(M) If R is any sub-region of the domain D for one of our functions the maximum (M′: minimum) value of the function in R cannot be reached at any interior point of R.

It is easily shown that postulates (1), (2), (3) and (M) characterize the continuous sub-harmonic functions. Similarly (1), (2), (3) and (M′) characterize the continuous super-harmonic functions.

[4] l. c. p. 43, p. 49, p. 50.
[5] F. Riesz, l. c. p. 331.

3. **The general case.** To formulate a set of postulates which will apply to the discontinuous case, we must replace (1) above by the weaker

(1A) Each function is semi-upper continuous in the n variables at all interior points of its domain D, and never becomes infinite faster than $1/r^{n-2}$ ($\log r$ for $n=2$), r being the distance from the point in question.

We retain postulates (2), (3), (4) without change.

To compensate for the weakened continuity requirement, we must replace (5) by the more stringent

(5A) Let a given n-dimensional sphere lie entirely inside the domain D belonging to $u(x_i)$, a function of the class. Let u be the limit of the difference of two decreasing series of continuous functions, $v_k(x_i)=a_k-b_k$. Let u_j and v_{jk} be further functions obtained from u and v_k by a finite number of rotations about x_{oi}, the center of the sphere, and taking of averages. If for points on the surface of the sphere, x_{si}, $\lim_{j=\infty} v_{jk}(x_{si})=0$, the limit being approached uniformly over the surface, then the sequence of values of the functions $u_j(x_{oi})$ cannot approach a *positive* limit.

These five postulates characterize the sub-harmonic functions in the sense that every class of functions satisfying them is a class of sub-harmonic functions, and the widest class of functions satisfying them is the class consisting of the totality of sub-harmonic functions. On replacing "semi-upper continuous" by "semi-lower continuous" in (1A), and "*positive* limit" by "*negative* limit" in (5A) we obtain a set of postulates for the super-harmonic functions.

We shall again prove the sub-harmonic character of our functions by first establishing a mean value inequality. Let us, then, fix our attention on a particular sphere S, lying entirely inside of D, the domain of $u(x_i)$, and let x_{oi} be the center of the sphere.

By postulate (1A), $u(x_i)$ is semi-upper continuous so that there exists a series of continuous functions $v_k(x_i)$, decreasing as k increases, having $u(x_i)$ as their limit. We form a series of decreasing positive constants, ϵ_j such that $\lim_{j=\infty} \epsilon_j=0$. For each j, we form an "average" function $u_j(x_i)$, as before, but based on the uniform continuity of the first j $v_k(x_i)$. We apply the same averaging process to the $v_k(x_i)$, to obtain functions $v_{jk}(x_i)$. The $u_j(x_i)$ will be functions of the class, by the first four postulates, and we

may also bring it about that

$$|v_{jh}(x_i) - \text{av } v_h(x_i) \text{ over } S| \leq \epsilon_j, \text{ for } h \leq j. \tag{6}$$

From the averaging process, we will have

$$v_{jk}(x_{oi}) = v_k(x_{oi}), \text{ for all } j, \tag{7}$$

and

$$u_j(x_{oi}) = u(x_{oi}). \tag{8}$$

We now wish to apply postulate (5A) to the sequence of functions

$$U_j(x_i) = u_j(x_i) - \text{av } u \text{ over } S, \tag{9}$$

and will show that we may take as the sequence of continuous functions approaching $U_j(x_i)$, for k becoming infinite,

$$V_{jk} = v_{jk}(x_i) - \text{av } v_k \text{ over } S. \tag{10}$$

Since $u(x_i)$ is the limit of the sequence $v_k(x_i)$, from the nature of the averaging process it follows that $u_j(x_i)$ is the limit of the sequence $v_{jk}(x_i)$. For the averages, we first note that if $E_k(\eta)$ is the $n-1$ dimensional measure of points in the $n-1$ dimensional set T_k for which

$$|\mu(x_i) - v_k(x_i)| \geq \eta, \text{ if } u(x_i) \text{ is finite} \tag{11}$$

and

$$|V_k(x_i)| \leq 1/\eta, \text{ if } u(x_i) \text{ is negatively infinite,} \tag{12}$$

it can be made less than δ, for $k \geq N(\eta, \delta)$. For, if this were not so, there would be an infinite number of values of k for which the measure $E_k \geq \delta$. As each set E_k includes the set E_{k+1}, since the v_k decrease to u, there would be a set of measure $\geq \delta$, and hence containing at least one point, common to all the E_k. But, at this point, the v_k could not approach u as a limit.

We now select an arbitrary positive δ and divide the points of S into three sets. The first set, the same for all k, S_1 is composed of $n-1$ dimensional regions including all the points of S at which $u(x_j)$ becomes (negatively) infinite, taken so small that u is negative throughout S_1 (and hence, for sufficiently great k, $\geq N'$, v_k is also negative there) and such that the integral of u over S_1 is numerically less than $\delta A/3$, A being the measure of S. This is possible, as postulate (1A) requires the infinities of u to be isolated

ones, integrable on S. Next select the $N(\delta/3, A\,\delta/3M)$, M being an upper bound for $|u|$ on the set of points $S-S_1$. For $k \geq N, N'$, let S_2 consist of the points in T_k, *i.e.*, satisfying (11) and (12), which are not in S_1. Thus (11), with right member $\delta/3$ does not hold in the set $S-S_1-S_2$, which we take as S_3.

We wish to obtain bounds for

$$\text{av } u - \text{av } v_k \text{ (over } S) = \frac{1}{A}\int_S (u - v_k)\, dS. \tag{13}$$

Computing the contributions from S_1, S_2 and S_3 separately, we find:

$$\left|\frac{1}{A}\int_{S_1}\right| \leq \frac{1}{A}\cdot\frac{\delta A}{3} \leq \frac{\delta}{3}, \quad k > N' \tag{14}$$

since the integral of u exceeds that of $(u-v_k)$ for k N';

$$\left|\frac{1}{A}\int_{S_2}\right| \leq \frac{1}{A}\cdot\frac{A\delta}{3M}\cdot M \leq \frac{\delta}{3}, \quad k > N,\ N'; \tag{15}$$

and

$$\left|\frac{1}{A}\int_{S_3}\right| \leq \frac{1}{A}\cdot\frac{\delta}{3}\cdot A \leq \frac{\delta}{3}, \quad k > N,\ N'. \tag{16}$$

Combining the last three inequalities, we have

$$|\text{av } u - \text{av } v_k| \leq \delta, \qquad k > N,\ N'. \tag{17}$$

As δ is arbitrary, this shows that, for $k = \infty$, the left member approaches zero, and av u is the limit of the sequence av v_k.

This shows that the sequence V_{jk} approaches U_j, as stated above. But, from (6), we see that $\lim_{j=\infty} V_{jk}(x_{ji}) = 0$, so that all the conditions of postulate (5) are met, and applying the conclusion of it, recalling (8) and (9), we find

$$\lim\ U_j(x_{oi}) = u(x_{oi}) - \text{av } u \text{ over } S \leq 0, \tag{18}$$

which is the mean value inequality we were seeking to establish.

The general property stated in the introduction (2) follows at once from this, for if $u - U \leq 0$ on the boundary of D' did not imply the same throughout D, $(u - U)$, as a semi-upper continuous

function in D would have to take on its maximum value G, necessarily positive, at some interior point of D. But, combining the mean value equality for the harmonic function U, with the mean value inequality (18) which we have established for u, we deduce that

$$u - U = G > 0 \tag{19}$$

on the boundary of any sphere entirely in D. On taking a sphere with a point in common with D', we are led to a contradiction.

COUCHE RANKS IN THE GENERAL MATRIX

By Lepine Hall Rice

In the *general matrix* of any number of directions and of any orders in the several directions, various kinds of rank have been defined and examined.[1] My contribution to the subject concerns the simpler ranks and brings out a certain relation among them. The proof flows from structural characteristics of the matrix, which must first be established. Let us begin with 3-way and 4-way matrices and lead from what is known or easily seen to what it is difficult but desirable to realize.

In a 3-way matrix there are three sets of parallel files, which are 1-way matrices of elements, running respectively in the three directions; and there are three sets of parallel layers, which are 2-way matrices of elements, lying respectively perpendicular to the three sets of files. Two layers of different aspects have in common, or intersect in, the elements of a file. Three layers of different aspects intersect in a single element.

Passing to a 4-way matrix, we find 1-way files of each of the four directions, and 3-way layers of each of the four corresponding aspects. Any layer is perpendicular to the files of one direction and contains files of the other three directions. Two layers of different aspects intersect in a 2-way couche. Three layers of different aspects intersect in a file. Four layers of different aspects intersect in a single element. The couche in which two of these four layers intersect and the couche in which the other two intersect are called contraspective couches, and with regard to either couche the other is called a counterlying couche. Layers and files are also called couches,[2] and a layer and a file that are mutually perpendicular are contraspective couches. Any two contraspective couches, then, intersect in a single element.

[1] Frank L. Hitchcock, *Multiple invariants and generalized rank of a p-way matrix or tensor* (this Journal, Vol. VII, No. 1, pp. 40–79, November 1927).

[2] The term *couche* is due to M. Lecat and is used in his treatise entitled *Coup d'Œil sur la Théorie des Déterminants Supérieurs* (Lamertin, Brussels, 1927), see p. 6.

Still considering a 4-way matrix, let us use indices in describing the above relations. Denote the matrix by

$$M = \|a_{\alpha\beta\gamma\delta}\| = (\alpha\beta\gamma\delta).$$

A layer of M is obtained by giving one index a fixed value while allowing the other three indices to take independently all their possible values, or, shortly, by holding fast one index while leaving free the remaining indices. That an index is held fast, may be denoted by placing a bar over it. Thus $(\bar{\alpha}\beta\gamma\delta)$ denotes a layer of aspect α, or an α-layer. An α-layer $(\bar{\alpha}\beta\gamma\delta)$ and a β-layer $(\alpha\bar{\beta}\gamma\delta)$ intersect in a 2-way couche $(\bar{\alpha}\bar{\beta}\gamma\delta)$ of aspect $\alpha\beta$ and direction $\gamma\delta$, or an $\alpha\beta$-couche. A γ-layer $(\alpha\beta\bar{\gamma}\delta)$ and a δ-layer $(\alpha\beta\gamma\bar{\delta})$ similarly intersect in a $\gamma\delta$-couche $(\alpha\beta\bar{\gamma}\bar{\delta})$. These two contraspective couches intersect in the single element $(\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta})$.

The elements of the matrix M can be rearranged to form a 2-way rectangular matrix[3] whose rows are the $\alpha\beta$-couches and whose columns are the $\gamma\delta$-couches, by placing in the same row all elements in whose indices α and β have the same values, and in the same column all elements in whose indices γ and δ have the same values. Or again, the elements can be rearranged to form a 2-way rectangular matrix whose rows are the α-layers of M and whose columns are the (counterlying) $\beta\gamma\delta$-couches (files of direction α). Such auxiliary rectangular matrices, formed with respect to the couches of any aspect and the counterlying couches, are of use in the treatment of rank in the general matrix, to which we now turn.

In a p-way matrix A, let U denote a selection of one or more indices and V denote the remaining indices. A U-couche is obtained by holding fast the index or indices U while leaving free the index or indices V. The U-couches and the V-couches are contraspective; in the extreme cases they are mutually perpendicular layers and files. The elements of A can be rearranged to form a 2-way rectangular matrix whose rows are the U-couches and whose columns are the V-couches. This matrix enables us to take over relations of linear dependence and rank from the 2-way theory and justifies the following treatment.

One and only one of the elements of each U-couche lies in each

[3] Hitchcock, *l. c.*

V-couche, and vice versa. We define *corresponding* elements of the couches of any aspect to be elements of the same counterlying couche.

Any k chosen couches of a given aspect are *linearly dependent* if there exist k constants, not all zero, such that when the couches are multiplied respectively by the constants the sum of each k corresponding elements is zero; otherwise the couches are linearly independent. If k couches are linearly dependent, the elements of at least one couche can be expressed as the same linear homogeneous functions of the corresponding elements of the other $k-1$ couches, and such a couche is said to be linearly dependent on the other $k-1$ couches. It can be reduced to zero by adding to it proper multiples of the other couches. Any k couches are linearly dependent if any one of them is zero.

The *U-rank* of a matrix is r if the matrix contains r linearly independent U-couches while every $r+1$ U-couches are linearly dependent. The U-rank and the V-rank are equal.

An *elementary transformation* of the U-couches of a matrix consists in either (*a*) interchanging two couches, or (*b*) multiplying a couche by a constant not zero, or (*c*) adding to (or subtracting from) a couche any multiple of another couche. An elementary transformation of the U-couches does not change either the U-rank or the V-rank of the matrix.

Theorem I. In the general matrix, an elementary transformation of the layers of any aspect leaves all couche-ranks unchanged.

Proof. First consider the 3-way matrix $(\alpha\beta\gamma)$. An α-layer is composed of $\alpha\beta$-couches (files of direction γ); it is also composed of $\alpha\gamma$-couches (files of direction β). Corresponding elements of the α-layers are evidently corresponding elements of certain $\alpha\beta$-couches, and are corresponding elements of certain $\alpha\gamma$-couches; *e.g.*, the elements a_{111} and a_{211} are corresponding elements of the $\alpha\beta$-couches (11γ) and (21γ), and are corresponding elements of the $\alpha\gamma$-couches $(1\beta1)$ and $(2\beta1)$. Hence an elementary transformation of the α-layers effects a number of elementary transformations of the $\alpha\beta$-couches and of the $\alpha\gamma$-couches. It therefore leaves the α-rank, the $\alpha\beta$-rank, and the $\alpha\gamma$-rank unchanged; and in consequence it leaves the $\beta\gamma$-rank, the γ-rank, and the β-rank

unchanged; and these are all the couche-ranks of the matrix. The same thing happens in the 4-way matrix $(\alpha\beta\gamma\delta)$. Here an α-layer is composed, according as we choose to regard it, of couches of aspect $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, $\alpha\beta\gamma$, $\alpha\beta\delta$, or $\alpha\gamma\delta$. Corresponding elements of α-layers are corresponding elements of certain couches of any one of these aspects. Therefore an elementary transformation of the α-layers leaves unchanged the α, $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, $\alpha\beta\gamma$, $\alpha\beta\delta$, and $\alpha\gamma\delta$ ranks, and in consequence it leaves unchanged the $\beta\gamma\delta$, $\gamma\delta$, $\beta\delta$, $\beta\gamma$, δ, γ, and β ranks, that is, all the couche-ranks of the matrix. And the same thing happens in the general matrix: an elementary transformation of the layers of any aspect π effects a number of elementary transformations of the couches of each aspect whose indices include π, therefore leaves unchanged the couche-ranks of these couches, and consequently leaves unchanged every other couche-rank.

Theorem II. The rank of the general matrix on the couches of any aspect cannot exceed the product of its ranks on the layers which intersect in these couches.

Proof. Denote the aspects of the layers by $u_1, u_2, \cdots, u_s$, and the aspect of the couches by $U = u_1u_2\cdots u_s$, and let the ranks be $r_1, r_2, \cdots, r_s$ and r_U. By elementary transformations of the u_1-layers we can reduce all but r_1 of them to zero while leaving all ranks unchanged. Operating in the same manner on the u_2-layers, we can reduce all but r_2 of them to zero while leaving all ranks unchanged. Moreover this will leave unchanged all the zeros previously made, since each element now operated on is operated on with elements in the same u_1-layer. Inasmuch as corresponding elements of layers of any aspect lie in the same layer of each other aspect, we can continue to operate without losing any of the zeros made, until we obtain finally a matrix in which there are, at most, $r_1r_2\cdots r_s$ non-zero intersections of s layers of the several aspects, that is, at most, $r_1r_2\cdots r_s$ non-zero U-couches. Therefore $r_U \leqq r_1r_2\cdots r_s$.

ANALYTIC AND TOPOLOGICAL TRANSFORMATIONS OF CLOSED SURFACES

By EDWARD SAIBEL

It has been shown by Franklin and Wiener[1] that every continuous one-to-one transformation of a two dimensional region of finite connectivity may be approximated to an arbitrary degree of exactness by an analytic one-to-one transformation. It is the object of this paper to show that this theorem can be extended to closed surfaces. Since many of the details go over without essential change, the author assumes a knowledge of the paper cited.

We have given a closed, analytic surface of genus p embedded in Euclidean three-space, and a transformation T, continuous and one-to-one which maps the given surface into itself or into a similar such surface. We wish to show that it is possible to approximate to the given topological transformation with an arbitrary degree of exactness by a one-to-one analytic transformation.

By an analytic transformation we mean one which is analytic in terms of two variables defined by a set of parametric curves on the surface, and we apply the usual connotation to analytic surface namely one whose equation is Cartesian coördinates is analytic.

Since the general closed surface of genus p is homeomorphic with the special closed analytic surface of genus p such as we use in the proof of our theorem, the parametric curves on the special surface will determine parametric curves on the general surface. Thus the operations which we perform on the special surface being defined in terms of these parameters, carry over to the general surface and are defined in terms of the parameters on it. This enables us to state the more general theorem: Given any closed surface of genus p and a transformation T continuous and one-to-one, which maps this surface into itself it is possible to approximate to T as closely as we please by a one-to-one analytic transformation.

A closed analytic surface of genus p can be constructed algebraically as follows. In Euclidean three-space let $G=O$ be the

[1]Franklin and Wiener, Trans. Amer. Math. Soc., Vol. 28, No. 4.

equation of a sphere and $C_1=O$, $C_2=O, \cdots C_p=O$ be the equations of p cylinders which pierce the sphere but which remain mutually exclusive, then with the proper convention of signs to obtain the surface inside the sphere, $G \cdot C_1 \cdot C_2 \cdots C_p=\epsilon$ will be the equation of a surface containing as a part a closed analytic surface of genus p, ϵ being a small positive quantity.

We shall first set up a network of convex curvilinear polygons on each of our surfaces. To each polygon on the one surface there will correspond an unique one on the other surface, and any transformation T_1 which takes one set of polygons into the other set will be a distance ϵ away from the transformation T. That being done, we construct two-parameter families of differentiable curves on both surfaces and by means of them set up the new transformation T_2 which is differentiable and one-to-one and which maps one set of polygons into the corresponding set. T_2 will be a distance ϵ away from T. Finally we shall use the Weierstrass approximation formula to obtain the desired transformation T_3, which is analytic, one-to-one, and satisfies the conditions of the theorem.

Corresponding networks of convex polygons on the surfaces. Let our two surfaces be s and S respectively. Triangulate surface s into arbitrarily small cells, the one cells of which are geodesic lines on the surface[3].

Let the length of the shortest side be σ. From each node measure off the distance $\sigma/3$ along the sides. Join the points thus formed about a node by geodesics. Cut out the lines inside of these newly formed figures and we have our surface mapped by a system of convex curvilinear polygons whose sides are geodesics. Three of these polygons meet at a node.

By a convex polygon on the surface we mean one any of whose interior points can be joined by a geodesic on the surface which does not intersect its boundary.

Transformation T carries this convex polygonal network into some irregular network on S.

Let $[A]$ be the set of all distances from a vertex of a polygon on s to another vertex of the same polygon. Let η be the least upper bound of the set $[A]$, and let η be so chosen that if on s

[3]Veblen, Cambridge Coll. 1916, part II, pp. 41–42.

$[p_1-p_2]<\eta$, then on S $[P_1-P_2]<H<\frac{\epsilon}{3}$, where $[M-N]$ denotes the shortest distance from M to N on the surface.

We now divide up s and S simultaneously to obtain corresponding convex polygons whose sides are geodesics. To do this, we first set up a correspondence between the periphery of a polygon on s and a possibly reëntrant polygon on S. This is done in essentially the same manner as in the plane case.[3] We surround each node of the transformed network on S by a polar geodesic circle and everywhere use geodesic segments in place of straight lines. This gives us a polygonal network on S corresponding to the convex polygonal network on s. The distance on s from a point in a mesh to a point in an abutting mesh is at most 2η. If $\overline{T}_1$ is any continuous one-to-one transformation taking each mesh of the network on s into the corresponding mesh of the polygonal network just set up on S, we shall have

$$\text{maximum} \quad [\mathrm{T}(p)-\overline{T}_1(p)]<2H.$$

The inverse transformation $\overline{T}_1^{-1}$ must take a point on S into the same mesh that T^{-1} does, or at least into a mesh abutting on it, so that we have

$$\text{maximum} \quad [\mathrm{T}^{-1}(P)-\overline{T}_1^{-1}(P)]\leq 2\eta<2H.$$

And as in the plane case by use of the inverse transformation T^{-1} we get a correspondence between the periphery of a polygon on s and a possibly reëntrant polygon on S.

The notion of visibility carries over to the surface if we substitute geodesic for straight line. Then as in the plane case we proceed to simultaneously divide up the corresponding polygons on s and S to obtain similar-placed triangles whose sides are geodesics. We then form corresponding convex polygons whose sides are geodesic segments.[4] Three of these polygons will meet at a node.

Let T_1 be a continuous (1, 1) transformation taking each convex polygon on s into its corresponding convex polygon on S. Points on S arising from a given point on s under $\overline{T}_1$ and T_1 respectively,

[3] Franklin and Wiener, *loc. cit.*, p. 765.
[4] Franklin and Wiener, *loc. cit.*, p. 766.

come from the same mesh or at worst from adjacent meshes under $\overline{T}_1$. Consequently under T these points come from meshes at worst next but one, or points on s at distance less than 3η.

$$\text{Maximum } [T(p) - T_1(p)] < 3H < \epsilon.$$

Similarly the inverse transformations T_1^{-1} and T^{-1} take a point on S into two points on s in at worst next but one meshes.

$$\text{Maximum } [T^{-1}(p) - T_1^{-1}(P)] < 3\eta < 3H < \epsilon.$$

Hence, the distance between T_1 and T is less than ϵ.

The differentiable transformation. The next stage in this process is the setting up of the differentiable transformation that will map one set of polygons on to the other set.

At each node of the network three lines meet, forming convex sectors. These must be transformed into three similar such lines. Consider any node, let it be the origin of a system of plane geodesic coördinates, and let the three lines meeting at it be the lines $t=0$, $t=\tan^{-1}(m)$, and $t=\tan^{-1}(-n)$.
We have to transform

$$\begin{array}{lll} t=0 & \text{into} & T=0 \\ t=\tan^{-1}(m) & \text{into} & T=\tan^{-1}(M) \\ t=\tan^{-1}(-n) & \text{into} & T=\tan^{-1}(-N) \end{array}$$

where m, n, M, N, are positive. We may use the transformation

$$A \equiv \left\{ \begin{array}{l} T=\tan^{-1}\left[\dfrac{MN(m+n)\sin t}{mn(M+N)\cos t+(mN-nM)\sin t}\right] \\ R=k\,F(M,\ N,\ m,\ n,\ r,\ t) \end{array} \right\},$$

where k is chosen small enough to map the three vertices adjacent to the origin on s onto the interior of the corresponding sides on S respectively. The Jacobian of this transformation is

$$J\left(\frac{t,r}{T,R}\right) = \frac{r}{R}\,k^2 mnMN(M+N)\ (m+n) > 0.$$

This is an affine transformation defined in the neighborhood of every vertex of the polygonal network.

Surround each vertex on s by a triangle which lies close to the vertex. Let the sides of the triangle be geodesic segments orthogonal to the sides of the network and for the moment, we shall define the differentiable transformation inside and on the boundary of the triangle to be the affine transformation A. This transformation carries the triangle on s into some curvilinear triangle on S about the corresponding node. Let n_1 and n_2 be two consecutive vertices of the network on s and let N_1 and N_2 be the corresponding vertices on S. Consider the side n_1n_2. We embed this side in a field of geodesics of which it is a member. Let this field of geodesics extend inside of the triangles about each node, although they need not extend up to the node. Call this family $[t]$. In a similar manner we embed side N_1N_2 on the other surface S in a field of differentiable curves of which it is a member, and we let these curves extend a little ways inside of the curvilinear triangles about N_1 and N_2 respectively. This family we shall designate as $[T]$. The only restriction we shall impose on $[T]$ is that none of the curves shall be tangent to the sides of the curvilinear triangle. The set $[t]$ will be the $t=$const. curves in our coördinate system in the neighborhood of a side of the network on s. Similarly $[T]$ will form the $T=$const. curves on S. We shall designate n_1n_2 as $t=0$ and its transform N_1N_2 as $T=0$. We shall form the $u=$const. curves by linear interpolation. We let the side of the triangle near n_1 be $u=0$ and choose the scale so that the side near n_2 will be $u=1$. We shall fill in the $u=$const. curves as follows; measure off the distance u_0 along $t=0$ from $u=0$. Let the length of any particular $t=$const. curve be l, the point along this curve $u=u_0$ will be a distance lu_0 from $u=0$ measured along this particular $t=$const. curve. In a similar manner we fill in the $U=$const. curves, after having designated the side of the curvilinear triangle near N_1 as $U=0$ and that near N_2 as $U=1$. So much for the coördinate system in the neighborhood of a side of the network between the sides of the triangles.

We shall now set up a coördinate system inside of the triangles As the $u=$const. curves we shall use a set of differentiable curves orthogonal to n_1n_2 and close to $u=0$ or $u=1$ as the case may be. Let the $U=$const. curves inside a triangle on S be the transforms of $u=$const. under the affine transformation A. The $T=$const.

curves will be the extensions inside the triangle of the family $[T]$ used above, and their transforms under A the curves $t=\text{const.}$

Let the Cartesian coördinates in which the surfaces s and S are expressed be (x, y) and (X, Y) respectively. We have tacitly assumed that the families of curves above have been so chosen that the Jacobians $J\left(\frac{u, t}{x, y}\right)$ and $J\left(\frac{U, T}{X, Y}\right)$ are greater than zero. We also note that the u and t curves are differentiable also a (u, t) gives a single point on the surface; similarly the U and T curves are differentiable and a (U, T) gives a single point on the surface and conversely. Thus the coördinates we have set up inside of the triangles are admissible.

Since a single (x, y) on s is given by a (u, t) and conversely and a single (X, Y) on S is given by a (U, T) and conversely, and the (u, t), (U, T) curves are differentiable also $J\left(\frac{u, t}{x, y}\right)>0$ and $J\left(\frac{U, T}{X, Y}\right)>0$, then a (1, 1) relation of (x, y) to (X, Y) with non-vanishing Jacobian implies a (1, 1) relation of (u, t) to (U, T) for which $J\left(\frac{u, t}{U, T}\right)$ does not vanish.[5] This means that our differentiable transformation which we have defined inside of the triangles. is (1, 1), and possesses a non-vanishing Jacobian.

Let $u=0$ be the outer boundary of a strip inside of which A is valid, and let the inner boundary of this strip be designated as $u=1$. Let their transforms $U=0$ and $U=1$ be the outer and inner boundaries of a strip inside of which the (U, T) coördinates are admissible.

In these coördinates the transformation A has the form

$$u=U$$

$$t=t\,(T, U)$$

The Jacobian of the transformation is

$$J\left(\frac{u, t}{U, T}\right)=\frac{\partial t}{\partial T}>0.$$

[5] Franklin and Wiener, *loc. cit.*, p. 767.

Inside of the strip $u=1$ and the triangle we define the differentiable transformation T_2 to be A. Between $u=1$ and $u=0$ we use a transformation derived from A which agrees with A up to the first derivatives at $u=1$ and which takes the curves $t=$const. into curves tangent to the curves $T=$const. at $u=0$. This transformation is

$$A_1 \equiv \begin{cases} u = U \\ t = t\,(T,\ 2U^2 - U^3) = t_1(T, U). \end{cases}$$

A_1 agrees with A at $u=1$ up to the first derivatives. At $u=0$, $\left(\frac{\partial t_1}{\partial U}\right)_{U=0} = 0$ while $\left(\frac{\partial t_1}{\partial T}\right)_{U=0} > 0$. It is (1, 1) for values of u or U between 0 and 1. Since $u = U$ and any value of $u = U$ in this range substituted into $t = t(T,\ U)$ gives a (1, 1) relation of t to T, it follows that $t = t(T,\ U)$ is (1, 1) for these values. But $2U^2 - U^3$ takes on the values between 0 and 1 when U does. Hence for a fixed U or u, $t = t,\ (T,\ U)$ gives a (1, 1) relationship of t to T. The Jacobian of the transformation is $\frac{\partial t_1}{\partial T} = \frac{\partial t}{\partial T} > 0$.

The next stage is to tack on a differentiable transformation which maps the side of the network and its neighborhood over on to the side corresponding on the other surface and its neighborhood. This transformation must join on to the affine transformation at both ends. We have already established corresponding coördinate systems along corresponding sides, which are made up of differentiable curves. These formed an admissible system of coördinates. This transformation we are now seeking must be of the form $u = u(U)$, $t = t(T)$ because of the manner in which the correspondence between the neighborhoods was set up. This transformation must satisfy the following conditions:

$$0 = u(0);$$

$$1 = u(1);$$

and have

$$\frac{\partial u}{\partial U} = 1;\ \frac{\partial u}{\partial T} = 0;$$

it must further take $t=0$ into $T=0$ and have

$$\frac{\partial t}{\partial U}=0 \text{ at } U=0 \text{ and } U=1;$$

$$\frac{\partial t}{\partial T}>0 \text{ at } U=0 \text{ and } U=1;$$

$u=U$ satisfies all the requirements on u. To define a $t(T)$ satisfying the above conditions, we make use of the following auxiliary function $h(T)$.

$$h(T)=1+\frac{T(M-1)}{k} \; ; \quad 0\leq T\leq k;$$

$$h(T)=M \; ; \quad k\leq T\leq V-k;$$

$$h(T)=E+\frac{(V-T)(M-E)}{k} \; ; \quad V-k\leq T\leq V;$$

where $M=\dfrac{v-\frac{1}{2}(1+E)}{V-k}$ and $k<\dfrac{V}{2}$, $k<\dfrac{2v}{1+E}$;

$$E(T)=\left.\frac{\partial t}{\partial T}\right|_{T=V}.$$

We note that $h(T)$ is a straight line function positive throughout and that the area under it is v. Also $h(0)=1$. Consequently if we put

$$t(T)=\int_0^T h(T)dT,$$

we see that $\left.\dfrac{\partial t}{\partial U}\right|_{U=0}=0$ and $\left.\dfrac{\partial t}{\partial U}\right|_{U=1}=0$, while $\dfrac{\partial t}{\partial T}=h(T)$

is positive throughout, also $t\ (0)=0$. The differentiable transformation is therefore defined to be

$$B\equiv\begin{cases} u=U \\ t=\displaystyle\int_0^T h(T)dT \end{cases}$$

along the sides. The Jacobian is $J\left(\dfrac{u,\,t}{U,\,T}\right)=h(T)>0$.

To interpolate inside of the polygon we draw a closed curve close to the boundary of the polygon and inside of it. We can form this curve of one of the $t=$ const. curves between $u=0$ and $u=1$ of the coördinates used in the neighborhood of a side of the polygon, and of segments of a polar geodesic circle in the triangular regions. This curve will be differentiable and if we take it close enough to the boundary of the polygon, will be visible from some inside point. Its transform under the transformations A, A_1, and B will be a differentiable curve inside of and close to the boundary of the corresponding polygon on S, and will be visible from some interior point. We proceed just as in the plane case to set up a system of modified polar coördinates and later of polar geodesic coördinates and in an analogous manner to form the differentiable transformation which is (1, 1), maps the interior of a polygon on the one surface on to the interior of the corresponding polygon on the other surface, and joins on up to the first derivatives to the differentiable transformations we have set up in the neighborhood of a node, and along a side.[6] We have now a differentiable transformation for each polygon given in terms of an admissible coördinate system and this transformation is valid in a region including the polygon. Hence these transformations may be expressed in a particular set of Cartesian coördinates.[7] The totality of these gives us the differentiable transformation T_2 of the type we have been seeking.

The analytic approximation to the differentiable transformation. Before we can apply the Weierstrass approximation formula to our differentiable transformation T_2, we must make use of a lemma which will guarantee our analytic formula to be one-to-one. It being quite possible that everything be (1, 1) *im Kleinen* but when we piece the parts together some of them might overlap. The lemma we shall make use of is as follows: If $X=X(x, y)$, $Y=Y(x, y)$ is a continuous (1, 1) transformation, holding in the closed convex region s, for which X and Y possess first partial derivatives with respect to x and y continuous in s and for which the Jacobian of X and Y with respect to x and y never vanishes in s,

[6] Franklin and Wiener, *loc. cit.*, pp. 772–776.
[7] Franklin and Wiener, *loc. cit.*, p. 768.

and $X = X_a(x, y)$, $Y = Y_a(x, y)$ are a pair of differentiable functions which approximate X and Y and whose first partial derivatives approximate those of X and Y uniformly in s, then, if the approximation is sufficiently close, these equations represent a one-to-one transformation for the interior of s.[8]

By means of this lemma we can show that any differentiable transformation that approximates to our T_2 with a sufficient degree of accuracy, both as to coördinates and their derivatives, will give a (1, 1) transformation of the closed analytic surface of genus p into itself or into a similar such surface. The proof follows exactly the proof given for the sphere[9] except as before we use for circle, polar geodesic circle, and when we project the neighborhood of a point on to the tangent plane at that point we project along the normals to the surface.

If we have a transformation mapping a portion of space which includes s in its interior onto a portion of space which includes S in its interior such that s goes into a surface S' close to S, we can draw S' into S along the normals to S in a (1, 1) analytic manner providing S' is close enough to S so that the normals to surface S do not intersect one another.

Let the differentiable transformation T_2 be given in terms of the three differentiable functions $X_2(x, y, z)$, $Y_2(x, y, z)$, and $Z_2(x, y, z)$. By the use of the Weierstrass approximation formula in three variables

$$f_k(x, y, z) =$$

$$\frac{1}{k^3\pi^{3/2}} \int\int\int_R f(X, Y, Z) e^{-\frac{(X-x)^2}{k^2} - \frac{(Y-y)^2}{k^2} - \frac{(Z-z)^2}{k^2}} dX\, dY\, dZ,$$

we can obtain the three analytic functions $X_3(x, y, z)$, $Y_3(x, y, z)$, and $Z_3(x, y, z)$ which define the analytic transformation T_3. It can be shown that if $f(x, y, z)$ in the above formula possesses continuous first partial derivatives over any closed region D interior to R, we have

[8] Franklin and Wiener, *loc. cit.*, p. 777.
[9] *Ibid.*, pp. 782–784.

$$\lim_{k\to\infty} f_k(x, y, z) = f(x, y, z);$$

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial x} = \frac{\partial f}{\partial x};$$

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial y} = \frac{\partial f}{\partial y};$$

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial z} = \frac{\partial f}{\partial z}.$$

If we take our approximation by the Weierstrass formula close enough, the analytic transformation defined by the three analytic functions $X_3(x, y, z)$, $Y_3(x, y, z)$, and $Z_3(x, y, z)$ which we have designated by T_3, will map s into a surface very close to S. Then we can map this surface on to S in an analytic (1, 1) manner by projection along the normals to S. Since this analytic transformation can be made as close as we please to the differentiable transformation, it lies within a distance ϵ of our original topological transformation T.

Special cases. Case 1, $p=0$. The discussion of this case by Franklin and Wiener, *loc. cit.*, is slightly incomplete in that objections might be taken as to whether the sphere can be divided up into networks of convex polygons. The process indicated above, clearly resolves this matter in the affirmative.

Case 2, $p=1$. The surface of the torus can be put into (1, 1) correspondence with a rectangle in the plane and the process used in the proof of the theorem for the plane case can be carried over directly if special precaution is taken along the edges of the rectangle, so that when the rectangle is laid back again on the torus, the edges will join up.

The equation of the torus is usually given in terms of the two angles ϕ and θ. The network on the torus will consist of the curves $\phi=$const. and $\theta=$const. Thus when the surface of the torus is rolled out on the plane this network goes into a rectangular network covering a rectangle r. We map the surface of the torus s onto r and its eight adjacent rectangles. Transformation T which is continuous and (1, 1) maps s on to S another torus which

we roll in the same manner on to a rectangle R and its eight adjacent rectangles. The edges of S will run over into these adjacent rectangles and if we are careful to do exactly the same thing to opposite edges, we can build up the convex polygonal network and the differentiable transformation just as was done in the plane case. We can then roll both networks back on to their respective surfaces and the edges will join up. Also the differentiable transformation will fit along the edges. Then we can use the Weierstrass approximation as was done for the sphere, and the theorem follows.

The author wishes to express his indebtedness to Prof. Philip Franklin for his kind assistance with this paper.

COHERENCY MATRICES AND QUANTUM THEORY

By NORBERT WIENER

Introduction

The statistical method of determining the linear relations between a number of coördinates recorded in a sequence of simultaneous observations is the theory of correlation coefficients,[1] which may be cast into a matrix form. This method, however, pays no attention to the order of the observations constituting the sequence. Attempts have been made to apply the method of correlation to time-series, using such notions as that of the correlation of a function with itself under a time-lag.[2] These methods, while they have achieved a certain measure of success, are largely of an *ad hoc* nature, and as yet have failed to crystallize out any simple, general theory.

It is the purpose of the present paper to develop a theory which seems to the author to possess more of an intrinsic logical necessity than those already existing. This theory is also applied to the study of polarized light and of quantum phenomena. It is the conviction of the author that it explains the linear character of the Schrödinger equation, and that it serves as an important link between the Heisenberg and Schrödinger theories, the classical mechanics, as modified by the introduction of Kaluza's fifth dimension, and the Bohr quantum rules. The author reserves a more detailed discussion of these relations for a later paper.

§1. **Definition of the Coherency Matrix.** We start with a set of (possibly complex valued) functions $f_1(t)$, $f_2(t)$, $\cdots$, $f_n(t)$, all defined over the infinite range $-\infty < t < \infty$. We make the definition

$$\phi_{jk}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_j(t+\tau)\overline{f}_k(t)dt. \tag{1}$$

Here, as throughout the sequel, the barring of a quantity denotes the formation of the conjugate complex quantity. We suppose that $\phi_{jk}(\tau)$ exists for $0 < j, k \leq n$, $-\infty < \tau < \infty$.

[1] Cf. G. U. Yule, *An Introduction to the Theory of Statistics* (London, 1919), Chapter IX-XII.

[2] Cf. F. C. Mills, *Statistical Methods* (New York, 1924), Chapter XI.

As we show in the appendix,

$$\phi_{kj}(\tau) = -\overline{\phi}_{jk}(\tau).$$

We also show that all the functions $\phi_{jk}(\tau)$ are bounded.

We now introduce the new definition

$$R_{jk}(u) = \int_{-1}^{1} \phi_{jk}(\tau) \frac{e^{iu\tau}-1}{i\tau} d\tau + \underset{A\to\infty}{\text{l.m.}} \left[\int_{1}^{A} + \int_{-A}^{-1}\right] \phi_{jk}(\tau) \frac{e^{iu\tau}}{i\tau} d\tau. \tag{2}$$

Formally this is equivalent to

$$R'_{jk}(u) = \int_{-\infty}^{\infty} \phi_{jk}(\tau) e^{iu\tau} d\tau,$$

but the theory of the integrated function is more general. Of course, this definition only determines $R_{jk}(u)$ except for a set of values of u of measure zero. To make the definition more precise, we shall take $R_{jk}(u)$ to be the derivative of its integral wherever the latter exists, and this, by a well-known theorem, will be everywhere over the region where $R_{jk}(u)$ exists except for a set of points of measure zero. At other points, $R_{jk}(u)$ will be undefined. $R_{jk}(u-0)$ and $R_{jk}(u+0)$, however, may still have an unambiguous definition.

The function $R_{jk}(u)$ will always exist and be summable and of summable square over any finite range, as $(e^{iu\tau}-1)i\tau$ is finite near $\tau=0$, while $\phi_{jk}(\tau)/i\tau$ is quadratically summable over any infinite interval not including the origin. $R_{jk}(u)$ will in general be complex even when the $f_k(u)$'s are real, since $e^{iu\tau}$ is complex. We shall however have

$$\begin{aligned} R_{kj}(u) &= \int_{-1}^{1} \phi_{kj}(\tau) \frac{e^{iu\tau}-1}{i\tau} d\tau + \underset{A\to\infty}{\text{l.m.}} \left[\int_{1}^{A} + \int_{-A}^{-1}\right] \phi_{kj}(\tau) \frac{e^{iu\tau}}{i\tau} d\tau \\ &= \int_{-1}^{1} \overline{\phi}_{jk}(-\tau) \frac{e^{iu\tau}-1}{i\tau} d\tau + \underset{A\to\infty}{\text{l.m.}} \left[\int_{1}^{A} + \int_{-A}^{-1}\right] \overline{\phi}_{jk}(-\tau) \frac{e^{iu\tau}}{i\tau} d\tau \\ &= \int_{-1}^{1} \overline{\phi}_{jk}(\sigma) \frac{e^{-iu\sigma}-1}{-i\sigma} d\sigma + \underset{A\to\infty}{\text{l.m.}} \left[\int_{1}^{A} + \int_{-A}^{-1}\right] \overline{\phi}_{jk}(\sigma) \frac{e^{-iu\sigma}}{-i\sigma} d\sigma \\ &= \overline{R}_{jk}(u). \end{aligned} \tag{3}$$

In particular, $R_{jj}(u)$ is real. The matrix $\|R_{jk}(u+0)-R_{jk}(u-0)\|$ is Hermitian. We call it a *coherency matrix*.

To see the meaning of $R_{jk}(u)$, let us consider the particular case where

$$f_j(t)=\sum_1^m A_{jh}e^{i\Lambda_h t}. \quad [\Lambda_1<\Lambda_2<\cdots<\Lambda_m]$$

Then

$$\phi_{jk}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{h,\,0=1}^{m}A_{jh}\overline{A}_{kl}e^{i\Lambda_h\tau}e^{i(\Lambda_h-\Lambda_e)t}dt$$

$$=\sum_{h=1}^{m}A_{jh}\overline{A}_{kh}e^{i\Lambda_h\tau};$$

$$R_{jk}(u)=\int_{-1}^{1}\sum_{h=1}^{m}A_{jh}\overline{A}_{kh}\frac{e^{i\tau(u+\Lambda_h)}-e^{i\tau\Lambda_h}}{i\tau}d\tau$$

$$+\left[\int_1^\infty+\int_{-\infty}^{-1}\right]\sum_{h=1}^{m}A_{jh}\overline{A}_{kh}\frac{e^{i\tau(u+\Lambda_h)}}{i\tau}d\tau$$

$$=\sum_{h=1}^{m}A_{jh}\overline{A}_{kh}\left[K+\int_{-\infty}^{\infty}\frac{\sin\tau(u+\Lambda_h)}{\tau}d\tau\right], \tag{4}$$

where K is some constant. Thus

$$R_{jk}(v)-R_{jk}(u)=\pi\sum_{h=1}^{m}A_{jh}\overline{A}_{kh}[\operatorname{sgn}(v+\Lambda_h)-\operatorname{sgn}(u+\Lambda_h)]$$

$$=2\pi\sum_{h=p}^{h=q}A_{jh}\overline{A}_{kh}, \tag{5}$$

provided that

$$\cdots<\Lambda_{p-1}<-v<\Lambda_p<\cdots<\Lambda_q<-u<\Lambda_{q+1}<\cdots$$

In particular,

$$R_{jk}(-\Lambda_h+0)-R_{jk}(-\Lambda_h-0)=2\pi A_{jh}\overline{A}_{kh}$$

so that the coherency matrix $\| R_{jk}(-\Lambda_k+0)-R_{jk}(-\Lambda_k-0)\|$, or

$$2\pi \begin{Vmatrix} |A_{1k}|^2 & A_{2k}\bar{A}_{1k} & \cdots \\ A_{1k}\bar{A}_{2k} & |A_{2k}|^2 & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Vmatrix}$$

is manifestly Hermitian.

In the case where the $f_j(t)$'s are trigonometrical polynomials, the diagonal terms of $\| R_{jk}(v)-R_{jk}(u)\|$ indicate the sum of the squares of the amplitudes belonging to frequencies of the f_j's lying between $-u$ and $-v$. They are, roughly speaking, of the nature of energies. The non-diagonal terms may be regarded in the light of mutual energies, such as we find recorded in an integrating wattmeter when we superimpose a current from one source on an electromotive force from another. We shall find these ideas suggestive even when the f_j's are not trigonometrical polynomials.

§2. **Transformations of Coherency Matrices.** We now introduce the idea of a linear transformation of the coördinates $f_j(t)$, of the nature of what Weyl[3] calls a *unitary transformation*. Let

$$g_j(t)=\sum_1^n a_{jk}f_k(t), \tag{6}$$

where

$$\sum_{k=1}^n a_{jk}\bar{a}_{hk}=S_{jh}\,. \qquad [1\leq j,\, h\leq n] \tag{7}$$

Let us put

$$\psi_{jk}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_j(t+\tau)\bar{g}_k(t)dt;$$

$$S_{jk}(u)=\int_{-1}^{1}\psi_{jk}(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

$$+\operatorname*{l.m.}_{A\to\infty}\left[\int_1^A+\int_{-A}^{-1}\right]\psi_{jk}(\tau)\,\frac{e^{iu\tau}}{i\tau}\,d\tau.$$

[3] H. Weyl, *Quantenmechanik und Gruppentheorie*, Zs. für Physik 46 (1927), p. 4.

Then

$$\psi_{jk}(\tau) = \sum_{h,\, l=1}^{n} a_{jh}\bar{a}_{kl}\phi_{hl}(\tau);$$

$$S_{jk}(u) = \sum_{h,\, l} a_{jh}\bar{a}_{kl}R_{hl}(u).$$

From the Hermitian matrix

$$\| R_{jk}(v) - R_{jk}(u) \|$$

let us construct the Hermitian form

$$\sum_{jk}[R_{jk}(v) - R_{jk}(u)]x_j\bar{x}_k.$$

The corresponding form derived from $S_{jk}(u)$ will be

$$\sum_{jkhl}[R_{hl}(v) - R_{hl}(u)]a_{jh}x_j\bar{a}_{kl}\bar{x}_k$$

$$= \sum_{jk}[R_{jk}(v) - R_{jk}(u)]y_j\bar{y}_k,$$

where

$$y_j = \sum_h a_{hj}x_h.$$

That is, the vector y_j is derived from the vector x_j by a unitary transformation, and the S-form is transformed into the R-form by the inverse of this transformation, which is likewise unitary.

By a well-known theorem,[4] we may choose our unitary transformation so that

$$S_{jk}(v) - S_{jk}(u) = 0. \qquad [j \neq k]$$

We shall suppose the a_{hj} so chosen as to effect this result. By the results of a previous paper of the author,[5] $S_{jj}(u)$ is monotone increasing, and the matrix $\|S_{jk}(v) - S_{jk}(u)\|$ assumes the form

$$\begin{Vmatrix} A & 0 & 0 & \cdots \\ 0 & B & 0 & \cdots \\ 0 & 0 & C & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix}$$

[4] H. Weyl, *loc. cit.*, p. 44.

[5] Norbert Wiener, *The Spectrum of an Arbitrary Function*, Proc. London Math. Soc. 27 (1928), p. 489.

where $A, B, C, \cdots$ are non-negative if $v \geq u$. The functions $S_{jk}(u)$ are also bounded. Since the form $Ax_1\bar{x}_1 + Bx_2\bar{x}_2 + \cdots$ is non-negative, if we transform our diagonal matrix into

$$\left\| \begin{array}{ccc} A_{11} & A_{21} & \cdots \\ A_{12} & A_{22} & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\|$$

by a unitary transformation, we shall always have

$$\sum_{jk} A_{jk} x_j \bar{x}_k \geq 0$$

and the converse will hold. We shall call such an Hermitian form *positive definite*. The sum of two such forms will clearly be positive definite.

In the case of such a form, if we make all x_h's vanish except x_j and x_k, and then remember that the determinant of a matrix is an invariant under a unitary transformation, we see that

$$|A_{jk}|^2 \leq A_{jj} A_{kk}.$$

A fortiori,

$$|A_{jk}| \leq \frac{A_{jj} + A_{kk}}{2}.$$

It hence results that the total variation of $R_{jk}(u)$ cannot exceed half the sum of the total variations of $R_{jj}(u)$ and $R_{kk}(u)$. As these latter functions are bounded and monotone, they are of limited total variation, and $R_{jk}(u)$ is likewise of limited total variation. It is therefore continuous except for a denumerable set of finite jumps, and possesses a derivative except for a set of points of zero measure. Thus the matrix

$$\left\| \begin{array}{ccc} R'_{11}(u) & R'_{21}(u) & \cdots \\ R'_{12}(u) & R'_{22}(u) & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\|$$

exists for almost all values of u, and may also be regarded as a coherency matrix. It is easy to see that it is positive definite, as it represents a limit of positive definite matrices.

§3. **Examples of Coherency Matrices.** We give below some examples of second order coherency matrices. We choose matrices of the form $\| R_{jk}(\Lambda+0) - R_{jk}(\Lambda-0) \|$, as these are simpler to illustrate. We shall write for purposes of abbreviation

$$F(x) = \frac{\sqrt{2}\cos \Lambda x}{\sqrt{\pi}};$$

$$G(x) = \frac{\sqrt{2}\cos \Lambda (x+\sqrt{|x|})}{\sqrt{\pi}}.$$

We put the functions f_1 and f_2 generating each matrix above the matrix it generates:

$$f_1(t) = F(t); \quad f_2(t) = aF(t+\xi).$$

$$\begin{Vmatrix} 1 & ae^{i\Lambda\xi} \\ ae^{-i\Lambda\xi} & a^2 \end{Vmatrix} \tag{A}$$

$$f_1(t) = aF(t); \quad f_2(t) = bG(t).$$

$$\begin{Vmatrix} a^2 & 0 \\ 0 & b^2 \end{Vmatrix} \tag{B}$$

$$f_1(t) = aF(t); \quad f_2(t) = bF(t+\xi) + cG(t).$$

$$\begin{Vmatrix} a^2 & abe^{i\Lambda\xi} \\ abe^{-i\Lambda\xi} & b^2+c^2 \end{Vmatrix} \tag{C}$$

§4. **Incoherent Addition of Matrices.** As a particular case of the effect of a linear transformation on a matrix, let us form the coherency matrix of $f_1(t), \cdots, f_n(t), g_1(t), \cdots, g_m(t)$ for a particular u, and let it have the form

$$\left\| \begin{array}{cccccccc} A_{11} & A_{21} & \cdots & A_{n1} & 0 & 0 & \cdots & 0 \\ A_{12} & A_{22} & \cdots & A_{n2} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{1n} & A_{2n} & \cdots & A_{nn} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & B_{11} & B_{21} & \cdots & B_{n1} \\ 0 & 0 & \cdots & 0 & B_{12} & B_{22} & \cdots & B_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & B_{1n} & B_{2n} & \cdots & B_{nn} \end{array} \right\|.$$

Then the coherency matrix of $f_1(t)+g_1(t)$, $f_2(t)+g_2(t)$, $\cdots$, $f_n(t)+g_n(t)$ will have the form

$$\left\| \begin{array}{cccc} A_{11}+B_{11} & A_{21}+B_{21} & \cdots & A_{n1}+B_{n1} \\ A_{12}+B_{12} & A_{22}+B_{22} & \cdots & A_{n2}+B_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{1n}+B_{1n} & A_{2n}+B_{2n} & \cdots & A_{nn}+B_{nn} \end{array} \right\|.$$

That is, if two phenomena are completely incoherent, the coherency matrix of their sum will be the sum of their coherency matrices.

§5. **Coherency Matrices and Polarized Light.** The theory of polarized light forms the simplest application of coherency matrices. The electric displacement in polarized light is a plane vector, a function of the time, which may be represented by two scalar functions of the time, its rectangular cartesian components. For each frequency, these yield a coherency matrix. A real unitary transformation of this matrix amounts simply to a rotation of our axes, together with a possible reflection. By such a transformation, the matrix may be brought into the form

$$\left\| \begin{array}{cc} A & -Bi \\ Bi & C \end{array} \right\|,$$

where

$$AC > B^2.$$

A matrix of this form may always be obtained by the incoherent addition of the two matrices

$$\begin{Vmatrix} \alpha^2 & -\alpha\beta i \\ \alpha\beta i & \beta^2 \end{Vmatrix}$$

and

$$\begin{Vmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{Vmatrix}.$$

The former represents completely elliptically polarized light, with the ellipse of polarization $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. The latter represents completely unpolarized light. The sum represents partly elliptically polarized light, with ellipse of polarization as above, percentage of polarization

$$\frac{\alpha^2+\beta^2}{100(\alpha^2+\beta^2+2\gamma^2)},$$

and total intensity $\alpha^2+\beta^2+2\gamma^2$.

As examples of simple types of polarized light, we may take

$$\begin{Vmatrix} 2 & 0 \\ 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 0 & 0 \\ 0 & 2 \end{Vmatrix}.$$

Plane polarized light at 0° and 90°.

$$\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix}.$$

Plane polarized light at 45° and 135°.

$$\begin{Vmatrix} 1 & -i \\ i & 1 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 1 & i \\ -i & 1 \end{Vmatrix}.$$

Circularly polarized light with opposite senses of rotation.

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Unpolarized light.

As is easily seen, the sum of the diagonal terms of a coherency matrix is an invariant under a unitary transformation. It represents the intensity of the light. Another invariant is the determinant of the matrix. If the latter vanishes, the matrix, if it does not vanish identically, assumes the form

$$\begin{Vmatrix} A & 0 \\ 0 & 0 \end{Vmatrix}$$

when reduced to diagonal form. Such a matrix will be called *completely polarized*. Two coherency matrices reducing to

$$\begin{Vmatrix} A & 0 \\ 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 0 & 0 \\ 0 & B \end{Vmatrix}$$

under the same unitary transformations will be said to be complementary. Thus a Nicol prism analyses light into two complementary components; circularly polarized light is complementary to polarized light with an opposite sense of rotation, etc. The analysis of light in terms of two complementary types is performed as follows: its matrix is transformed by the unitary transformation reducing the latter to diagonal form, and the terms on the principal diagonals represent the relative intensities of the two components.

§6. **Coherency Matrices and Quantum Theory.** We now come to the applications of coherency matrices to quantum theory. Our coherency matrices are here considered with reference, not to time, but with respect to the fifth dimension x_0 of Kaluza.[6] Moreover, instead of depending on an index j assuming a finite set of values, our functions of x_0 whose coherency we investigate contain the four continuous dimensions of space and time as parameters. That is, our original functions have the form $f(x, y, z, t; x_0)$. The functions $\phi_{jk}(\tau)$ are replaced by expressions of the form

$$\phi(x, y, z, t; x_1, y_1, z_1, t_1; \tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x, y, z, t; x_0+\tau)\, \bar{f}(x_1, y_1, z_1, t_1; x_0)dx_0$$

[6] Th. Kaluza, *Sitzungsber. der Berliner Akad.* 1921, p. 966.

while the $R_{jk}(u)$ are replaced by functions

$$R(x, y, z, t; x_1, y_1, z_1, t_1; u) = \int_{-1}^{1} \phi(x, y, z, t; x_1, y_1, z_1, t_1; \tau) \frac{e^{iu\tau} - 1}{i\tau} d\tau$$

$$+ \lim_{A \to \infty} \left[\int_{1}^{A} + \int_{-A}^{-1} \right] \phi(x, y, z, t; x_1, y_1, z_1, t_1; \tau) \frac{e^{iu\tau}}{i\tau} d\tau.$$

In other words, our Hermitian *matrices* are replaced by Hermitian *kernels* in the sense of Hilbert.[7]

Another matter of importance is that in the quantum theory only one single value $\frac{2\pi}{h}$ of the frequency u ever comes into play. In other words, our coherency kernel is of the form

$$\left[\frac{\partial R}{\partial u}(x, y, z, t; x_1, y_1, z_1, t_1, u) \right] = K(x, y, z, t; x_1, y_1, z_1, t_1).$$

We shall suppose this quantity K to exist.

In what follows, we shall make no attempt at mathematical rigor. To do so would require the entire theory of integral equations, non-singular and singular. To go into such intricacies would not serve in the last to illuminate the larger outlines of the quantum theory.

The problem of reducing an Hermitian kernel to diagonal form is already classical in the literature.[8] Under very extended hypotheses, the problem admits a unique solution, and may be treated by the aid of the so-called Hellinger integral. In the very simplest cases, the kernel is the kernel of a Fredholm equation, and may be expanded by the so-called bilinear formula in terms of the characteristic functions of the equation. These functions constitute a discrete set of normal and orthogonal functions, and the Hermitian kernel in the diagonal form is reduced to a matrix consisting of a discrete set of rows and columns. In the more general case, the discrete rows and columns of the diagonal matrix are supplemented by a continuous portion.

In general, the characteristic functions of the integral equation

[7] Cf. D. Hilbert, *Grundzüge einer Allgemeinen Theorie der linearen Integralgleichungen* (Leipzig and Berlin, 1912), pp. 162, 177.

[8] D. Hilbert, *loc. cit.*, p. 162.

$$\psi(x, y, z, t) = \lambda \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_1 \int_{\infty}^{\infty} dz_1$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} K(x, y, z, t; x_1, y_1, z_1, t_1) \psi(x_1, y_1, z_1, t_1) dt_1 \qquad (8)$$

do not depend in any peculiarly simple way on t. It is quite otherwise in the case when the coherency kernel is built up in the following manner: the function $f(x, y, z, t; x_0)$ is the sum of a large number of functions

$$f_1(x, y, z, t; x_0) + f_2(x, y, z, t; x_0) + \cdots$$

all having the same coherency kernel when referred to a proper time origin, and arranged at haphazard in time. When we reduce the coherency kernel of the sum function to a scale on which it is comparable to the coherency kernels of the partial functions composing it, it will clearly have the form

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} K(x, y, z, t+w; x_1, y_1, z_1, t_1+w) dw$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} K(x, y, z, t-t_1+\mu; x_1, y_1, z_1, \mu) d\mu$$

$$= Q(x, y, z; x_1, y_1, z_1; t-t_1).$$

If we expand this kernel trigonometrically in $t-t_1$, it will assume the form

$$\Sigma Q_k(x, y, z; x_1, y_1, z_1) e^{i\Lambda_k(t-t_1)}.$$

Here, of course, the sum may be wholly or partly replaced by an integral. If we now expand $Q_k(x, y, z; x_1, y_1, z_1)$ in terms of its characteristic functions, we have

$$Q_k(x, y, z; x_1, y_1, z_1) = \Sigma A_{kl} \psi_{kl}(x, y, z) \overline{\psi}_{kl}(x_1, y_1, z_1),$$

and

$$Q(x, y, z; x_1, y_1, z_1; t-t_1) = \sum_k \sum_l A_{kl} \psi_{kl}(x, y, z) e^{i\Lambda_k t} \overline{\psi}_{kl}(x_1, y_1, z_1) e^{-i\Lambda_k t_1}.$$

That is, the characteristic functions of the kernel $Q(x, y, z, t; x_1, y_1, z_1, t_1)$ are of the form

$$\psi_{kl}(x, y, z) e^{i\Lambda_k t},$$

and are orthogonal in a generalized sense over space-time. This is the same form as that of the characteristic functions of a steady-state Schrödinger equation.

If $f(x, y, z, t; x_0)$ is a solution of the five-dimensional Schrödinger equation, then $\phi(x, y, z, t; x_1, y_1, z_1, t_1; x_0)$ is also, as it is a mean of such solutions, displaced with respect to x_0, while this latter quantity does not occur in the coefficients of the five-dimensional Schrödinger equation. It follows readily that $K(x, y, z, t; x_1, y_1, z_1, t_1)$ is a solution of the four-dimensional Schrödinger equation. Hence

$$\psi_{kl}(x, y, z)e^{i\Lambda_k t} = \lambda \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dz_1$$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K(x, y, z, t; x_1, y_1, z_1, t_1)\psi_{kl}(x_1, y_1, z_1)e^{i\Lambda_k t_1} dt_1$$

is a solution of the same equation, and the characteristic functions of our coherency matrix are also characteristic functions of the Schrödinger equation, in the case where the coefficients do not contain the time explicitly. The exponents Λ_k are the corresponding characteristic numbers.

We thus see that in the case of a function satisfying Schrödinger's equation, the coherency kernel leads us back to Schrödinger's equation. On the other hand, any coherency kernel, even if it is not derived from a linear problem like that of the Schrödinger equation, gives rise to a linear integral equation determining a set of characteristic functions. This is not surprising, as the coherency matrices constitute a natural extension of correlation coefficients, while these latter quantities are only of use as a measure of the degree of *linear* dependence of one quantity on another. This suggests the conjecture that the linear character of the Schrödinger equation may be artificial; that it merely represents the characteristic function theory belonging to the coherency kernel of a problem of an essentially non-linear character. It may be that this is the sense in which it is possible to dispose of the wave-particle antinomy running through the whole of modern physics.

As an example of this possibility, let us reflect on a consideration put forward by Schrödinger himself in an early paper. He takes

$\phi_1(x_1, x_2, x_3, x_4)$, $\phi_2(x_1, x_2, x_3, x_4)$, $\phi_3(x_1, x_2, x_3, x_4)$, $\phi_4(x_1, x_2, x_3, x_4)$ as the components of the electromagnetic potential, and points out that if we choose a proper scale for our potentials

$$\int [\phi_1 dx_1 + \phi_2 dx_2 + \phi_3 dx_3 + \phi_4 dx_4]$$

over a period or quasiperiod will be an integral multiple of the Planck h when and only when the Bohr quantum condition is satisfied.[9] Now, the Kaluza five-dimensional quadratic form is

$$\sum_{j,k=1}^{4} g_{jk} dx_j dx_k + (\phi_1 dx_1 + \phi_2 dx_2 + \phi_3 dx_3 + \phi_4 dx_4 + dx_0)^2$$

and this vanishes for the paths of electrons. If our electron moves with a velocity low with respect to that of light, we may (with only a second order error) identify $\Sigma g_{jk} dx_j dx_k$ and $-dx_4^2$, and we get

$$dx_0 = -\phi_1 dx_1 - \phi_2 dx_2 - \phi_3 dx_3 - (\phi_4 - 1) dx_4.$$

If we choose a different value for our potential at infinity as the one belonging to the Schrödinger equation, we see that the change in x_0 over a period or quasiperiod is an integral multiple of h. It follows that if we put

$$x_1 = x_1(x_0), \cdots, x_4 = x_4(x_0)$$

the only permissible values of x_1, x_2, and x_3 are those containing $e^{\frac{2\pi i x_0}{h}}$ in their harmonic analysis.

If now we take the electron density over x_1, x_2, x_3, x_0 as our function $f(x_1, x_2, x_3, x_4, x_0)$, and build up its coherency kernel, no change in the number of those orbits not satisfying the Bohr frequency condition will affect this kernel. The characteristic numbers of the kernel under the hypothesis of steady state will not, however, represent the energies of the Bohr orbits, but their mechanical frequencies. We thus arrive at a characteristic function problem similar to that of the Schrödinger equation, but not identical with it. The nearness of our starting-point to that of the original Bohr theory is worthy of comment.

A word or two concerning the relation of our coherency kernels

[9] E. Schrödinger, Über eine bemerkenswerte Eigenschaft der Quantenbahnen eines einzelnen Elektrons. *Zs. für Physik* 12 (1923), pp. 13–24.

to the Heisenberg matrices may be in order. These latter matrices represent the transforms of the operators x, y, or z in the case of the q-matrices, $\frac{h}{2\pi i}\frac{\partial}{\partial x}$, $\frac{h}{2\pi i}\frac{\partial}{\partial y}$, or $\frac{h}{2\pi i}\frac{\partial}{\partial z}$ in the case of the p-matrices, by the transformations reducing our coherency kernel to diagonal form. It is interesting to pursue the analogy in the case of coherency matrices of the second order. In this case, the closest analogue to the x operator is the matrix

$$\left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|.$$

Then it may be verified that the Heisenberg matrix corresponding to the matrix

$$\left\| \begin{matrix} A & B-Ci \\ B+Ci & D \end{matrix} \right\|$$

is

$$\left\| \begin{matrix} \frac{A-D}{\sqrt{(A-D)^2+4(B^2+C^2)}} & \frac{2(B-Ci)}{\sqrt{(A-D)^2+4(B^2+C^2)}} \\ \frac{2(B+Ci)}{\sqrt{(A-D)^2+4(B^2+C^2)}} & \frac{D-A}{\sqrt{(A-D)^2+4(B^2+C^2)}} \end{matrix} \right\|$$

In particular, the Heisenberg matrices corresponding to plane polarized light at 0°, plane polarized light at 45°, and circularly polarized light in one direction, are

$$\left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|;$$

$$\left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|;$$

$$\left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\|.$$

It is perhaps worthy of comment that these three matrices play an important role in Weyl's treatment of the spinning electron by means of finite groups. If circularly polarized light or light polarized at 45° is analyzed by a Nicol prism at 0°, it appears unpolarized. The same remark applies if circularly polarized light or light polarized at 0° is examined by a Nicol prism at 45°, or if plane polarized light at 0° or at 45° is examined by a device which can only detect circular polarization. It is also true that light is completely characterized as to its state of polarization if it is analyzed by the instruments just mentioned and if its intensity is determined. This symmetry between the three sorts of light in question is purely formal and mathematical, and probably explains their relation to the Weyl group theory.

APPENDIX

By the Schwarz inequality,

$$|\phi_{jk}(\tau)| \leq \sqrt{\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f_j(t+\tau)|^2 dt \; \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{T}^{T} |f_k(t)|^2 dt}.$$

Now,

$$\begin{aligned}\overline{\lim_{T\to\infty}} \int_{-T}^{T} |f_j(t+\tau)|^2 dt &= \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T+\tau}^{T+\tau} |f_j(t)|^2 dt \\ &\leq \overline{\lim_{T\to\infty}} \frac{1}{2(T+\tau)}\int_{-T-\tau}^{T+\tau} |f_j(t)|^2 dt \\ &= \phi_{jj}(0).\end{aligned}$$

Therefore

$$|\phi_{jk}(\tau)| \leq \sqrt{\phi_{jj}(0)\phi_{kk}(0)}. \tag{1}$$

Thus the functions $\phi_{jk}(\tau)$ are all bounded.

Again,

$$\phi_{kj}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{f}_j(t) f_k(t+\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T+\tau}^{T+\tau} \bar{f}_j(t-\tau) f_{k(t)}\,dt$$

$$= \bar{\phi}_{jk}(-\tau) + \lim_{T\to\infty} \frac{1}{2T} \int_{T}^{T+\tau} \bar{f}_j(t-\tau) f_k(t)\,dt$$

$$+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T-\tau}^{-T} \bar{f}_j(t-\tau) f_k(t)\,dt. \tag{2}$$

By the Schwarz inequality,

$$\left| \lim_{T\to\infty} \frac{1}{2T} \int^{T+\tau} \bar{f}_j(t-\tau) f_k(t)\,dt \right| \leq \overline{\lim_{T\to\infty}} \frac{1}{2T} \sqrt{\int_{T-\tau}^{T} |f_j(t)|^2 dt \int_{T}^{T+\tau} |f_k(t)|^2 dt}$$

$$\leq \sqrt{\overline{\lim}\left[\frac{1}{2T}\int_{-T}^{T} |f_j(t)|^2 dt - \frac{1}{2(T-\tau)} \int_{-T+\tau}^{T-\tau} |f_j(t)|^2 dt\right] \frac{1}{2(T+\tau)} \int_{-T-\tau}^{T+\tau} |f_k(t)|^2 dt}$$

$$= 0. \tag{3}$$

Similarly,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+\tau}^{-T} \bar{f}_j(t-\tau) f_k(t)\,dt = 0. \tag{4}$$

Combining (2), (3), and (4), we get

$$\phi_{kj}(\tau) = \bar{\phi}_{jk}(-\tau). \tag{5}$$

A METHOD FOR DERIVING FORMULAS FOR THE APPROXIMATE CALCULATION OF INTEGRALS[1]

PRESCOTT D. CROUT

1. **Introduction.** The formulas which have been derived for the calculation of the approximate value of a proper integral are, for the most part, based upon the integral of a polynomial approximation of $f(x)$, the function to be integrated. Such formulas differ only in the data taken to fix the polynomial, and they are usually derived by considering either the polynomial itself,[2] or a Taylor's expansion[3] of $f(x)$. This expansion may be used to give expressions limiting the error,[4] and expressions for the error.[5]

The purpose of this paper is to develop a general method for deriving formulas of integration, together with general types of expressions which limit the error; and also to investigate the uniqueness and determinateness of results. The formulas of Cotes[6] (hence of Simpson and Villarceau), Gauss, Euler, and Tchebychef[7] will be derived, with particular expressions for the error, as special applications of the general theory.

Without loss of generality the origin will be placed at the center of the interval of integration, and $f(x)$ will be assumed to possess derivatives of such orders and in such intervals as are necessary to give meaning to the expressions which arise.

[1] This paper was prepared under the supervision of Prof. George Rutledge, of the Massachusetts Institute of Technology, who offered many valuable suggestions.

[2] *Goursat: Cours d'Analyse Mathématique*, v. 1, p. 266; also *Jordan: Cours d'Analyse*, v. 2, p. 125.

[3] *Lampe: Zur mechanischen Quadratur, Jahresbericht der Deutschen Mathematiker-Vereinigung*, v. 25 (1916), p. 325.

[4] *Peano: Residuo in formula de quadratura Cavalieri-Simpson, L'Enseignement Mathématique*, v. 18 (1916), p. 124; also *Tannery: Leçons d'algèbre et d'analyse*, v. 2, p. 555.

[5] *Bertrand: Traité de calcul différentiel et de calcul intégral*, v. 2, p. 331.

[6] *Rutledge: The Polynomial Determined by* $(2n+1)$ *Points, Journal of Mathematics and Physics*, v. 2 (1923), p. 47.

[7] *Radau: Etude sur les formules d'approximation servent à calculer la valeur numérique d'une intégrale définie, Journal de Mathématique*, s. 3, v. 6 (1880), p. 283.

2. **Formulas of Cotes.** By way of illustration let us derive a formula of integration involving

$$f_{-2}=f(-\theta),\ f_{-1}=f\left(-\frac{\theta}{2}\right),\ f_0=f(0),\ f_1=f\left(\frac{\theta}{2}\right),\ f_2=f(\theta).$$

Let $\phi(\theta)$ be defined by the expression

$$\phi(\theta)=\int_{-\theta}^{\theta}f(x)dx-\theta[K_{-2}f_{-2}+K_{-1}f_{-1}+K_0f_0+K_1f_1+K_2f_2], \quad (1)$$

where the K's are constants, and let us denote $\frac{d^if_j}{d\theta^i}$ by f_j^i. Using a special case of Leibnitz's rule for the derivative of a product, we have

$$\begin{aligned}
&\psi(\theta)=K\theta f(c\theta);\\
&\psi^1(\theta)=Kf(c\theta)\qquad\qquad +Kc\theta f^1(c\theta);\\
&\psi^2(\theta)=2Kcf^1(c\theta)\qquad +Kc^2\theta f^2(c\theta);\qquad (2)\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\psi^k(\theta)=kKc^{k-1}f^{k-1}(c\theta)+Kc^k\theta f^k(c\theta);
\end{aligned}$$

where K and c are constants. Using these relations to obtain successive derivatives of $\phi(\theta)$ we obtain the following:

$$\begin{aligned}
&\phi^1(\theta)=f_2+f_{-2}-K_{-2}f_{-2}-K_{-1}f_{-1}-K_0f_0\quad -K_1f_1-K_2f_2+\theta a_1;\\
&\phi^2(\theta)=f_2^1-f_{-2}^1+2K_{-2}f_{-2}^1+\frac{2}{2}K_{-1}f_{-1}^1\quad -\frac{2}{2}K_1f_1^1-2K_2f_2^1+\theta a_2;\\
&\phi^3(\theta)=f_2^2+f_{-2}^2-3K_{-2}f_{-2}^2-\frac{3}{4}K_{-1}f_{-1}^2\quad -\frac{3}{4}K_1f_1^2-3K_2f_2^2+\theta a_3;\quad (3)\\
&\phi^4(\theta)=f_2^3-f_{-2}^3+4K_{-2}f_{-2}^3+\frac{4}{8}K_{-1}f_{-1}^3\quad -\frac{4}{8}K_1f_1^3-4K_2f_2^3+\theta a_4;\\
&\phi^5(\theta)=f_2^4+f_{-2}^4-5K_{-2}f_{-2}^4-\frac{5}{16}K_{-1}f_{-1}^4\quad -\frac{5}{16}K_1f_1^4-5K_2f_2^4+\theta a_5;
\end{aligned}$$

where the θa's represent terms having θ as a factor. Letting $\theta\to 0$ and noting that when $\theta=0$,

$$f_j^i=f_0^i=f^i(0), \qquad (i=0,1,2,3,4;\ j=-2,-1,1,2.) \quad (4)$$

and $\qquad \theta a_{i+1}=0,$

we have

$$\phi^1(0) = [1+1-K_{-2}-K_{-1}-K_0-K_1-K_2]f_0$$

$$\phi^2(0) = [1-1+2K_{-2}+\frac{2}{2}K_{-1} \quad -\frac{2}{2}K_1-2K_2]f_0^1. \tag{5}$$

. . . . . .

For a reason to be made clear later, we determine the K's so as to have

$$0 = \phi^1(0) = \phi^2(0) = \phi^3(0) = \phi^4(0) = \phi^5(0) \tag{6}$$

regardless of f; hence from (5) we have

$$K_{-2}+K_{-1}+K_0+K_1+K_2 = 2;$$

$$2K_{-2}+\frac{2}{2}K_{-1} \quad -\frac{2}{2}K_1-2K_2 = 0.$$

. . . . . .

This system is determinate, for its determinant is of the Vandermonde type and hence not zero. Also, the work of computing the K's is reduced by noting that if $K_{-2} = K_2$ and $K_{-1} = K_1$; two equations are satisfied; two unknowns drop out; $\phi^i(0) = 0$, i even; and we have left three equations in three unknowns.[8] Solving this system gives the following result:

$$K_{-2} = K_2 = \frac{7}{45},\ K_{-1} = K_1 = \frac{32}{45},\ K_0 = \frac{12}{45}. \tag{7}$$

By the aid of (2) we now write

$$\phi^6(\theta) = f_2^5 - f_{-2}^5 + 6K_{-2}f_{-2}^5 + \frac{6}{32}K_{-1}f_{-1}^5 - \frac{6}{32}K_1f_1^5 - 6K_2f_2^5 \tag{8}$$

$$-\theta[K_{-2}f_{-2}^6 + \frac{1}{64}K_{-1}f_{-1}^6 + \frac{1}{64}K_1f_1^6 + K_2f_2^6];$$

[8] *Pascal: Die Determinanten*, p. 128, p. 204. Also, consider the equations due to $\phi^i(0) = 0$, i even; place $u = (K_{-2} - K_2)$, $v = (K_{-1} - K_1)$; and substitute. We have

$$2u + \frac{2}{2}v = 0$$

$$4u + \frac{4}{8}v = 0,$$

a system with determinant of the Vandermonde type; hence $u = v = 0$, or

$$K_{-2} = K_2, \quad K_{-1} = K_1.$$

which, after terms are collected, the symmetry of the K's noted, and the law of the mean applied, reduces to the following expression:

$$\phi^6(\theta) = (1-6K_{-2})(2\theta)f^6(\xi_1) - \frac{6}{32}K_{-1}\theta f^6(\xi_2) \tag{9}$$

$$-\theta[K_{-2}f_{-2}^6 + \frac{1}{64}K_{-1}f_{-1}^6 + \frac{1}{64}K_1 f_1^6 + K_2 f_2^6]$$

where $$-\theta < \xi_1 < \theta,$$

and $$-\frac{\theta}{2} < \xi_2 < \frac{\theta}{2}.$$

Let us denote the greatest $|f^6(x)|$ in the interval $(-\theta, \theta)$ by f_m^6. Then

$$|\phi^6(\theta)| \leq \theta f_m^6 \left\{ 2|1-6K_{-2}| + 2|K_{-2}| + \frac{7}{32}|K_{-1}| \right\}; \tag{10}$$

hence by (7), $$|\phi^6(\theta)| \leq \frac{3}{5}\theta f_m^6. \tag{11}$$

Also

$$|\phi^5(\theta)| = |\phi^5(\theta) - \phi^5(0)| = \left|\int_0^\theta \phi^6(\theta)d\theta\right| \leq \int_0^\theta |\phi^6(\theta)|\,d\theta \leq \frac{3\theta^2 f_m^6}{5\cdot 2}, \tag{12}$$

since $\phi^5(0)$ is zero. Noting (6) we repeat this process, with the final result:

$$|\phi(\theta)| \leq \frac{3\theta^7 f_m^6}{5\cdot 7!}. \tag{13}$$

Without the relations (6) this process would not be possible; hence the reason for so determining the K's is now evident. Substituting (7) in (1), and using (13) we have

$$\int_{-\theta}^{\theta} f(x)dx$$

$$= \frac{\theta}{45}\left\{ 7f(-\theta) + 32f\left(-\frac{\theta}{2}\right) + 12f(0) + 32f\left(\frac{\theta}{2}\right) + 7f(\theta) \right\} + \epsilon \tag{14}$$

where $$|\epsilon| \leq \frac{3\theta^7 f_m^6}{5\cdot 7!} = \frac{\theta^7 f_m^6}{8400},$$

which is the desired formula of integration.

We shall now consider the system of integration formulas based upon values of $f(x)$ at $(2n+1)$ *points which divide the interval* $(-\theta, \theta)$ *into* $2n$ *equal subintervals*. Adopting symmetrical notation we have, as in (1),

$$\phi(\theta) = \int_{-\theta}^{\theta} f(x)dx - \theta[K_{-n}f_{-n} + \cdots + K_0 f_0 + \cdots + K_n f_n] \qquad (15)$$

where $$f_i = f\left(\frac{i}{n}\theta\right).$$

Using (2), the relation (15) is differentiated $(2n+1)$ times with respect to θ. Placing $\theta = 0$, only the principal terms, those not of the form "θa," remain; hence the system of derivatives will be of the same form as (5). The K's are determined so as to have

$$\phi^1(0) = \phi^2(0) = \cdots = \phi^{2n+1}(0) = 0 \qquad (16)$$

regardless of f, the resulting system of equations being determinate since its determinant is of the Vandermonde type. The computations are simplified by noting, from (2) and (3), that principal terms involving negative subscripts (points to the left of the origin) alternate in sign during the course of the differentiations; whereas those with positive subscripts (points to the right of the origin) do not; consequently if

$$K_{-n} = K_n, \; K_{-(n-1)} = K_{(n-1)}, \cdots, \; K_{-1} = K_1, \qquad (17)$$

we have $$\phi^i(0) = 0, \; i \text{ even};$$

hence n relations of (16) are satisfied, and n unknowns fall out.[9] Using (2) it is a relatively simple matter to write down the general system that remains to be solved, the equations being of the form $\phi^i(0) = 0$, i odd. We have

$$2K_n + 2K_{n-1} + \cdots 2K_1 + K_0 = 2$$
$$kK_n + \cdots kK_i\left(\frac{i}{n}\right)^{k-1} + \cdots kK_1\left(\frac{1}{n}\right)^{k-1} = 1$$

where $k = 3, 5, 7, \cdots (2n+1)$ and in each equation $i = 1, 2, 3, \cdots n$.

[9] As before (see note [7]), the relations

$$\phi^i(0) = 0 \qquad i = 2, 4, 6, \cdots 2n$$

involving the unknowns: $u = (K_{-n} - K_n)$, $v = (K_{-(n-1)} - K_{(n-1)})$, $\cdots$ $w = (K_{-1} - K_1)$ may be considered separately. Since the determinant of this system is not zero, $u = v = \cdots w = 0$; hence (17) is obtained directly.

The first equation alone contains K_0, and merely states that $\sum_{i=-n}^{i=n} K_i = 2$; hence we need concern ourselves only with the n remaining equations. Multiplying the kth equation by $\frac{n^{k-1}}{k}$, we have the following n equations:

$$\sum_{i=1}^{i=n} \mathbf{K}_i \mathbf{i}^{k-1} = \frac{\mathbf{n}^{k-1}}{\mathbf{k}} \tag{18}$$

where $k = 3, 5, 7, \cdots (2n+1)$.

This system of equations yields Cotes' coefficients.[10]

We now place a limit upon the error, $\phi(\theta)$, by considering $\phi^{2n+2}(\theta)$. Noting (17) we collect terms and apply the law of the mean to terms involving f^{2n+1}; then placing

$$f_m^{2n+2} = \text{greatest } |f^{2n+2}(x)| \text{ in } (-\theta, \theta)$$

and again collecting terms, we have, using (2),

$$|\phi^{2n+2}(\theta)| \leq \sigma \theta f_m^{2n+2};$$

where $$\sigma = \left\{ 2|1-(2n+2)K_n| + (2n+2)\sum_{i=1}^{i=n-1} |K_i| \left(\frac{-}{n}\right)^{2n+1} \left(\frac{2i}{n}\right) + 2\sum_{i=1}^{i=n} |K_i| \left(\frac{i}{n}\right)^{2n+2} \right\}.$$

$$\therefore \sigma = \left\{ \mathbf{2}|\mathbf{1}-(\mathbf{2n}+\mathbf{2})\mathbf{K}_n| + \left[(\mathbf{4n}+\mathbf{6}) \sum_{i=1}^{i=n-1} |\mathbf{K}_i| \left(\frac{\mathbf{i}}{\mathbf{n}}\right)^{2n+2} \right] + \mathbf{2}|\mathbf{K}_n| \right\}. \tag{19}$$

Applying the reasoning of (12) repeatedly[11], noting (16), we have

$$|\phi(\theta)| \leq \frac{\sigma \theta^{2n+3} f_m^{2n+2}}{\mathbf{2n+3}!}. \tag{20}$$

In deriving the σ of (20) we have taken $\phi^{2n+2}(\theta)$ as the starting

[10] *Rutledge: The Explicit Determination of Cotes' Coefficients for Polynomial Area, Journal of Mathematics and Physics*, v. 1 (1922), p. 78. It should be noted that the K's of this reference are one half in value of the K's of (18).

[11] This is an extension and modification of a method that has been used to place a limit upon the error involved in the use of Simpson's rule: *Ch. J. de la Vallée Poussin: Cours d'analyse infinitésimale*, v. 1, p. 329; also *R. de Montessus et R. d'Adhémar: calcul numérique*, p. 166.

point. Obviously any derivative of lower order could have been taken, the even ones being preferable because their signs are such as to allow the application of the law of the mean to the principal terms to eliminate odd derivatives of f. Let $\phi^{2l}(\theta)$ be taken as the starting point; then, reasoning as above, we have

$$|\phi(\theta)| \leq \frac{\sigma_l \theta^{2l+1} f_m^{2l}}{2l+1!}, \text{ where } l \leq (n+1) \tag{21}$$

$$\text{and } \sigma_l = \left\{ 2\left|1-2l\mathbf{K}_n\right| + \left[(4l+2) \sum_{i=1}^{i=n-1} |\mathbf{K}_i| \left(\frac{i}{n}\right)^{2l} \right] + 2|\mathbf{K}_n| \right\}. \tag{22}$$

It should be noted in (19) and (22) that unless $n>1$, Σ drops out.[12] Placing the results of (18) in (15), we have, using (20) and (21),

$$\int_{-\theta}^{\theta} f(x)dx = \theta \left\{ K_{-n} f(-\theta) + \cdots + K_0 f(0) + \cdots + K_n f(\theta) \right\} + \epsilon$$

$$\text{where} \qquad |\epsilon| \leq \frac{\sigma_l \theta^{2l+1} f_m^{2l}}{2l+1!}, \qquad l \leq (n+1), \tag{23}$$

$$\text{or} \qquad |\epsilon| \leq \frac{\sigma \theta^{2n+3} f_m^{2n+2}}{2n+3!}.$$

The procedure in the case of the Cotes formulas based upon values of $f(x)$ at $2n$ points is identical with that above. The resulting system for determining the K's is

$$\sum_{i=1}^{i=n} K_i (2i-1)^{k-1} = \frac{(2n-1)^{k-1}}{k},$$

where $k=1, 3, 5, \cdots (2n-1)$. There is no K_0. We have

$$|\epsilon| \leq \frac{\sigma_l \theta^{2l+1} f_m^{2l}}{2l+1!}, \qquad l \leq n, \tag{24}$$

[12] Let $2n+1=3$, or $n=1$; then from (18) we have $K_{-1}=K_1=\frac{1}{3}$, $K_0=\frac{4}{3}$; from (19): $\sigma=\left\{2\left|1-\frac{4}{3}\right|+\frac{2}{3}\right\}=\frac{4}{3}$; from (22), $\sigma_1=\left\{2\left|1-\frac{2}{3}\right|+\frac{2}{3}\right\}=\frac{4}{3}$; and finally $\int_{-\theta}^{\theta} f(x)dx=\frac{\theta}{3}[f(-\theta)+4f(0)+f(\theta)]+\epsilon$, where $|\epsilon| \leq \frac{\sigma_1 \theta^3 f_m^3}{3!} = \frac{2\theta^3 f_m^3}{9}$, or $|\epsilon| \leq \frac{\sigma \theta^5 f_m^4}{5!} = \frac{\theta^5 f_m^4}{90}$, which is Simpson's rule for one division. Cf. footnote 11.

where

$$\sigma_l = \left\{ 2|1-2lK_n| + \left[(4l+2) \sum_{i=1}^{i=n-1} |K_i| \left(\frac{2i-1}{2n-1} \right)^{2l} \right] + 2|K_n| \right\}.$$

Comparing (23) and (24), we should expect the formulas based upon $(2n+1)$ points to be the more desirable.

3. **Formulas Based upon Values of f(x) at Any Number of Arbitrary Points.** It will now be shown that formulas of integration can be derived that are based upon values of $f(x)$ at any number of arbitrary points, within or without the interval $(-\theta, \theta)$. As in the preceding section we place

$$\phi(\theta) = \int_{-\theta}^{\theta} f(x)dx - \theta[K_1 f(c_1\theta) + K_2 f(c_2\theta) + \cdots K_n f(c_n\theta)] \qquad (25)$$

where the c's, which are distinct, are not restricted in either magnitude or sign; but once chosen are fixed, leaving the K's to be determined. Placing

$$\phi^1(0) = \phi^2(0) = \cdots = \phi^n(0) = 0,$$

we have, using (2), regardless of f, a system for determining the K's which has the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2c_1 & 2c_2 & 2c_3 & \cdots & 2c_n \\ 3c_1^2 & 3c_2^2 & 3c_3^2 & \cdots & 3c_n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ nc_1^{n-1} & nc_2^{n-1} & nc_3^{n-1} & \cdots & nc_n^{n-1} \end{vmatrix} = n! \prod_{\substack{i,j=1 \\ i>j}}^{n} (c_i - c_j)$$

which is of the Vandermonde[7] type and hence not zero, since the points are distinct. Thus the system is determinate.

A limit may be placed upon the error by considering any derivative of $\phi(\theta)$ of order equal to or smaller than $(n+1)$; however, since in this case the law of the mean cannot in general be applied to eliminate terms involving odd derivatives of $f(x)$, this limit will in general contain two terms.

As an application of this generalization we have the following formula of integration:

$$\int_{-\theta}^{\theta} f(x)dx = \frac{\theta}{12}\left\{ f(-2\theta)+22f(0)+f(2\theta) \right\} +\epsilon$$

where $$|\epsilon| \leq \frac{38\theta^6 f_m^4}{3\cdot 5!} = \frac{19\theta^5 f_m^4}{180}$$

and $f_m^4 =$ greatest $|f^4(x)|$ in the interval $(-2\theta, 2\theta)$.

4. **Formulas of Tchebychef.** Thus far we have considered the c's as being initially fixed, the K's representing coefficients to be determined; however, we now alter this point of view. Let

$$\phi(\theta) = \int_{-\theta}^{\theta} f(x)dx - \theta[Kf(c_{-n}\theta) + Kf(c_{-n+1}\theta) + \cdots Kf(c_n\theta)] \quad (26)$$

where $2n+1$ points are involved.

Proceeding as above, with the aid of (2), we differentiate successively and determine the unknown constants by the relations

$$\phi^1(0) = \phi^2(0) = \cdots \phi^{2n+2}(0) = 0,$$

which give the following system of $(2n+2)$ equations:

$$\begin{aligned} K \quad &+ K \quad &+ \cdots \quad K \quad &= 2 \\ 2Kc_{-n} &+ 2Kc_{-n+1} &+ \cdots 2Kc_n &= 0 \\ 3Kc^2_{-n} &+ 3Kc^2_{-n+1} &+ \cdots 3Kc^2_n &= 2 \\ 4Kc^3_{-n} &+ 4Kc^3_{-n+1} &+ \cdots 4Kc^3_n &= 0. \\ \cdot \quad &\cdot \quad &\cdot \quad &\cdot \\ \cdot \quad &\cdot \quad &\cdot \quad &\cdot \end{aligned} \quad (27)$$

It is evident that K could not have been fixed arbitrarily; also, we note that if

$$c_0 = 0 \text{ and } c_{-i} = -c_i, \quad i = 1, 2, 3, \cdots n,$$

then $(n+1)$ relations of (27) are satisfied, and $(n+1)$ unknowns fall out. We thus have

$$K = \frac{2}{2n+1}$$

$$\sum_{i=1}^{i=n} c_i^{k-1} = \frac{2n+1}{2k},$$

where $k=3, 5, 7, \cdots (2n+1)$, as the system to be solved. Taking $\phi^{2l}(\theta)$ as the basis for the limit upon $|\epsilon|$, we have

$$|\epsilon| \leq \frac{\sigma_l \theta^{2l+1} f_m^{2l}}{\underline{2l+1}\,!}, \tag{28}$$

where

$$\sigma_l = 2\left\{1+2\left(\frac{2l+1}{2n+1}\right)\sum_{i=1}^{i=n} c_i^{2l}\right\} \text{ and } l \leq (n+1);$$

also $c_i > 0$ if $i=1, 2, 3, \cdots n$; for the symmetry of the c's allows the use of the law of the mean, as with (8).

Had we started with $2n$ *values of* $f(x)$ *instead of* $(2n+1)$, we should arrive at the following result:

$$c_{-i} = -c_i \qquad \text{where } i=1, 2, \cdots n,$$

$$\mathbf{K} = \frac{\mathbf{1}}{\mathbf{n}},$$

$$\sum_{i=1}^{i=n} \mathbf{c}_i^{k-1} = \frac{\mathbf{n}}{\mathbf{k}},$$

where $k=3, 5, 7, \cdots (2n+1)$ (n equations).

Also

$$|\epsilon| \leq \frac{\sigma_l \theta^{2l+1} \mathbf{f}_m^{2l}}{\underline{\mathbf{2l+1}}\,!}, \tag{29}$$

where

$$\sigma_l = \mathbf{2}\left\{\mathbf{1}+\left(\frac{\mathbf{2l+1}}{\mathbf{n}}\right)\sum_{i=1}^{i=n} \mathbf{c}_i^{2l}\right\} \text{ and } l \leq (n+1).$$

Here $c_i > 0$ if $i=1, 2, 3, \cdots n$.

Comparing (28) and (29) we should expect the formulas based upon $2n$ values of $f(x)$ to be more desirable than those based upon $(2n+1)$ values.[13]

[13] This is the opposite of the conclusion in the case of the Cotes formulas. See (23) and (24).

As an example let $(2n+1)=5$ and $l=3$. Then

$$K=\frac{2}{5}, \qquad c_1^2+c_2^2=\frac{5}{6}, \qquad c_1=-c_{-1}=\sqrt{\frac{5-\sqrt{11}}{12}},$$

$$c_0=0, \qquad c_1^4+c_2^4=\frac{5}{10}, \qquad c_2=-c_{-2}=\sqrt{\frac{5+\sqrt{11}}{12}},$$

$$\int_{-\theta}^{\theta} f(x)dx=\frac{2\theta}{5}\left\{f(c_{-2}\theta)+f(c_{-1}\theta)+f(0)+f(c_1\theta)+f(c_2\theta)\right\}+\epsilon,$$

$$|\epsilon|\leq\frac{\sigma_3\theta^7 f_m^6}{7!} \text{ where } \sigma_3=\frac{419}{108}.$$

5. **Formulas of Gauss.** We shall now leave both the K's and the c's as unknown constants, and place

$$\phi(\theta)=\int_{-\theta}^{\theta} f(x)dx-\theta[K_{-n}f(c_{-n}\theta)+\cdots+K_n f(c_n\theta)]. \tag{30}$$

Using (2) we differentiate successively, and determine both the K's and the c's so that

$$\phi^1(0)=\phi^2(0)=\cdots=\phi^{2N}(0)=0,$$

where N is the total number of points; thus $N=2n$ or $2n+1$ according to whether N is even or odd. No difficulty is encountered in writing down the system to be solved:

$$\begin{aligned}
&K_{-n}+K_{-n+1}+\cdots+K_{n-1}+K_n=2\\
&2K_{-n}c_{-n}+2K_{-n+1}c_{-n+1}+\cdots+2K_{n-1}c_{n-1}+2K_n c_n=0\\
&3K_{-n}c^2_{-n}+3K_{-n+1}c^2_{-n+1}+\cdots+3K_{n-1}c^2_{n-1}+3K_n c^2_n=2\\
&4K_{-n}c^3_{-n}+4K_{-n+1}c^3_{-n+1}+\cdots+4K_{n-1}c^3_{n-1}+4K_n c^3_n=0\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{31}$$

$2N$ equations in all.

If $N=2n+1$, we note that if

$$c_0=0 \text{ and } \begin{matrix} c_{-i}=-c_i \\ K_{-i}=K_i \end{matrix} \text{ where } i=1, 2, 3, \cdots n,$$

the even equations of (31), N in all, are satisfied; and N unknowns drop out. We then have[3]

$$\mathbf{K}_0+2\sum_{i=1}^{i=n}\mathbf{K}_i=2,$$

$$\sum_{i=1}^{i=n}\mathbf{K}_i\mathbf{c}_i^{k-1}=\frac{1}{\mathbf{k}}, \tag{32}$$

where $k=3, 5, 7, \cdots (2N-1)$.

If $N=2n$, we note that if

$$\begin{matrix} c_{-i}=-c_i \\ K_{-i}=K_i \end{matrix} \quad \text{where } i=1, 2, 3, \cdots n,$$

the N even equations of (31) are satisfied; and N unknowns drop out. We then have

$$\sum_{i=1}^{i=n}\mathbf{K}_i=1,$$

$$\sum_{i=1}^{i=n}\mathbf{K}_i\mathbf{c}_i^{k-1}=\frac{1}{\mathbf{k}}, \tag{33}$$

where $k=3,\ 5,\ 7, \cdots (2N-1)$.

In either case, due to the symmetry of the c's we may use the law of the mean upon $\phi^{2l}(\theta)$ to eliminate odd derivatives of f, with the result that

$$|\epsilon| \leq \frac{\sigma_l \theta^{2l+1} f_m^{2l}}{2l+1!} \quad \text{where } 1 \leq N,$$

and $$\sigma_l=\left\{ 2+2\cdot 2^l \sum_{i=1}^{i=n} |K_i|\, c_i^{2l-1} c_i \qquad +2\sum_{i=1}^{i=n} |K_i| c_i^{2l} \right\},$$

$$\sigma_l=2\left\{ 1+(2l+1)\sum_{i=1}^{i=n} |\mathbf{K}_i|\mathbf{c}_i^{2l} \right\}. \tag{34}$$

As an example let $N=3$; then from (32) we have

$$K_0+2K_1=2,$$

$$K_1c_1^2=\frac{1}{3},$$

$$K_1c_1^4=\frac{1}{5},$$

which gives

$$c_0=0, \qquad c_1=\sqrt{\frac{3}{5}}=-c_{-1}, \qquad K_1=\frac{5}{9}=K_{-1}, \qquad K_0=\frac{8}{9}.$$

We also have, from (34),

$$\sigma_3=2\left\{1+7\left(\frac{3}{5}\right)^3\left(\frac{5}{9}\right)\right\}=\frac{92}{25},$$

$$\sigma_2=2\left\{1+5\left(\frac{3}{5}\right)^2\left(\frac{5}{9}\right)\right\}=4,$$

$$\sigma_1=2\left\{1+3\left(\frac{3}{5}\right)\left(\frac{5}{9}\right)\right\}=4.$$

Thus the desired formula is

$$\int_{-\theta}^{\theta}f(x)dx=\frac{\theta}{9}\left\{5f\left(-\theta\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\theta\sqrt{\frac{3}{5}}\right)\right\}+\epsilon$$

where

$$|\epsilon|\leq\frac{4\theta^3f_m^{\,2}}{3!}=\frac{2\theta^3f_m^{\,2}}{3},$$

$$|\epsilon|\leq\frac{4\theta^5f_m^{\,4}}{5!}=\frac{\theta^5f_m^{\,4}}{30},$$

$$|\epsilon|\leq\frac{92\theta^7f_m^{\,6}}{25\cdot7!}=\left(\frac{23}{25}\right)\frac{\theta^7f_m^{\,6}}{1260}.$$

6. **Formulas of Christoffel.** It is evident that certain K's and c's may be fixed,[14] those remaining being then determined in the usual manner. For example, by placing

$$\phi(\theta)=\int_{-\theta}^{\theta}f(x)dx-\theta\left[\frac{1}{10}f(-\theta)+K_{-1}f(c_{-1}\theta)+K_0f(0)+K_1f(c_1\theta)+\frac{1}{10}f(\theta)\right], \tag{35}$$

[14] *Christoffel: Uber die Gaussische Quadratur und eine Verallgemeinerung derselben, Journal für Mathematik*, v. 55 (1858), p. 61.

we find that

$$\int_{-\theta}^{\theta} f(x)dx$$

$$= \frac{\theta}{90}\left\{ 9f(-\theta)+49f\left(-\theta\sqrt{\frac{3}{7}}\right)+64f(0)+49f\left(\theta\sqrt{\frac{3}{7}}\right)+9f(\theta)\right\}+\epsilon$$

where $$|\epsilon| \leq \frac{8\theta^7 f_m^6}{5\cdot 7!}.$$

7. **Formulas Involving Higher Powers of θ.** Thus far, the formulas considered have been of the form

$$\int_{-\theta}^{\theta} f(x)dx = \theta \sum_{i=i_1}^{i=i_2} K_i f(c_i\theta) \qquad +\epsilon.$$

We shall now consider formulas of the more general type:

$$\int_{-\theta}^{\theta} f(x)dx = \theta \sum_{i=i_{11}}^{i=i_{12}} K_{i,1} f(c_{i,1}\theta)+\theta^2 \sum_{i=i_{21}}^{i=i_{22}} K_{i,2} f^1(c_{i,2}\theta)+\cdots+\epsilon, \tag{36}$$

or

$$\int_{-\theta}^{\theta} f(x)dx = \sum_{j=1}^{j=j_2} \theta^j \sum_{i=i_{j,1}}^{i=i_{j,2}} K_{i,j} f^{j-1}(c_{i,j}\theta) \qquad +\epsilon,$$

where $K_{i,j}$ and $c_{i,j}$ are constants independent of $f(x)$.

As an illustration let us take a specific case, placing

$$\phi(\theta) = \int_{-\theta}^{\theta} f(x)dx - \theta[A_{-2}f_{-2}+A_2 f_2]$$
$$-\theta^2[B_{-2}f_{-2}{}^1+B_{-1}f_{-1}{}^1+B_1 f_1{}^1+B_2 f_2{}^1]-\theta^3 C_0 f_0{}^2 \tag{37}$$

where A_{-2}, A_2, $B_{-2}\cdots B_2$, C_0 are constants; and the five points in question (those to which the subscripts refer) are such as to divide the interval $(-\theta, \theta)$ into four equal subintervals. For convenience in the differentiation that is to come, we note a special case of Leibnitz's rule:

$$\begin{aligned}
\psi(\theta) &= K\theta^2 f^1(c\theta)\\
\psi^1(\theta) &= 2K\theta f^1 + K\theta^2 c f^2\\
\psi^2(\theta) &= 2K f^1 + 4K\theta c f^2 + K\theta^2 c^2 f^3\\
\psi^3(\theta) &= 6Kc f^2 + 6K\theta c^2 f^3 + K\theta^2 c^3 f^4\\
\psi^4(\theta) &= 12Kc^2 f^3 + 8K\theta c^3 f^4 + K\theta^2 c^4 f^5\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\psi^8(\theta) &= 56Kc^8 f^7 + 16K\theta c^7 f^8 + K\theta^2 c^8 f^9.
\end{aligned} \tag{38}$$

We are interested only in the first column, except when placing a limit upon $|\epsilon|$, for the terms of the other columns are of the form "θa".

Using (2) and (38) we write

$$\phi^1(\theta) = +f_{-2}+f_2- A_{-2}f_{-2}-A_2f_2 \qquad +\theta a_1$$

$$\phi^2(\theta) = -f_{-2}^1+f_2^1+2A_{-2}f_{-2}^1-2A_2f_2^1-2B_{-2}f_{-2}^1-2B_{-1}f_{-1}^1-2B_1f_1^1 \qquad -2B_2f_2^1 \qquad +\theta a_2$$

$$\phi^3(\theta) = +f_{-2}^2+f_2^2-3A_{-2}f_{-2}^2-3A_2f_2^2+6B_{-2}f_{-2}^2 + \frac{6}{2}B_{-1}f_{-1}^2-\frac{6}{2}B_1f_1^2 \qquad -6B_2f_2^2-6C_0f_0^2 \qquad +\theta a_3$$

$$\phi^4(\theta) = -f_{-2}^3+f_2^3+4A_{-2}f_{-2}^3-4A_2f_2^3- 12\,B_{-2}f_{-2}^3-\frac{12}{4}B_{-1}f_{-1}^3 \qquad -\frac{12}{4}B_1f_1^3-12B_2f_2^3 \qquad +\theta a_4$$

$$\phi^5(\theta) = +f_{-2}^4+f_2^4-5A_{-2}f_{-2}^4-5A_2f_2^4+20B_{-2}f_{-2}^4+\frac{20}{8}B_{-1}f_{-1}^4 \qquad -\frac{20}{8}B_1f_1^4-20B_2f_2^4 \qquad +\theta a_5$$

$$\phi^6(\theta) = -f_{-2}^5+f_2^5+6A_{-2}f_{-2}^5-6A_7f_2^5-30B_{-2}f_{-2}^5-\frac{30}{16}B_{-1}f_{-1}^5 \qquad -\frac{30}{16}B_1f_1^5-30B_2f_2^5 \qquad +\theta a_6$$

$$\phi^7(\theta) = +f_{-2}^6+f_2^6-7A_{-2}f_{-2}^6-7A_2f_2^6+42B_{-2}f_{-2}^6+\frac{42}{32}B_{-1}f_{-1}^6 \qquad -\frac{42}{32}B_1f_1^6-42B_2f_2^6 \qquad +\theta a_7, \tag{39}$$

where the θa's merely represent terms having θ as a factor. Proceeding as we did in the case of (3) we determine A_{-2}, A_2, B_{-2}, $\cdots$, B_2, C_0 so as to have

$$\phi^1(0)=\phi^2(0)=\cdots=\phi^7(0)=0 \tag{40}$$

for any f, noting that if

$$A_{-2}=A_2, -B_{-2}=B_2, \quad \text{and} \quad -B_{-1}=B_1, \tag{41}$$

three equations are satisfied and three unknowns drop out. Thus the system reduces to the following:

$$\begin{aligned} A_2 &= 1, \\ 3A_2+3B_1+6B_2+3C_0 &= 1, \\ 5A_2+\frac{5}{2}B_1+20B_2 &= 1, \\ 7A_2+\frac{21}{16}B_1+42B_2 &= 1, \end{aligned}$$

from which it results that

$$A_{-2}=A_2=1, \quad B_{-2}=-B_2=\frac{13}{105}, B_{-1}=-B_1=\frac{64}{105}, C_0=\frac{20}{105}.$$

In order to place a limit upon $|\epsilon|$, we use (2) and (38) to write $\phi^5(\theta)$ in full. Due to symmetry and to (41) we can combine terms and then use the law of the mean to eliminate odd derivatives of f. We then have

$$\begin{aligned} -\phi^5(\theta) &= (8A_2-1+56B_2)(2\theta)f^5(\xi_1)+\frac{56}{64}B_1\theta f^5(\xi_2) \\ &+\theta(f_{-2}^5+f_2^5)(A_2+16B_2)+\frac{16}{128}B_1\theta(f_{-1}^5+f_1^5) \\ &+2B_2\theta^3 f^{10}(\xi_3)+\frac{B_1}{256}\theta^3 f^{10}(\xi_4) \end{aligned} \tag{42}$$

where

$$\begin{aligned} -\theta &< \xi_1, \ \xi_3 < \theta \\ -\frac{\theta}{2} &< \xi_2, \ \xi_4 < \frac{\theta}{2}, \end{aligned}$$

from which we proceed as in the case of (9). Thus

$$\begin{aligned} |\phi^5(\theta)| &\leq \theta f_m^5 \\ &\left\{ 2|8A_2-1+56B_2|+\frac{56}{64}|B_1|+2|A_2+16B_2|+\frac{32}{128}|B_1| \right\} \\ &+\theta^3 f_m^{10}\left\{ 2|B_2|+\frac{1}{256}|B_1| \right\}; \end{aligned} \tag{43}$$

hence $|\phi^5(\theta)| \leq \frac{292}{105}\theta f_m^5+\frac{1}{4}\theta^3 f_m^{10}$,

an expression of two terms. Reasoning as in (12), noting (40), we integrate successively and have finally

$$\int_{-\theta}^{\theta} f(x)dx = \theta\{f_{-2}+f_2\} + \frac{\theta^2}{105}\{13f_{-2}{}^1+64f_{-1}{}^1+20\theta f_0{}^2-64f_1{}^1-13f_2{}^1\}+\epsilon \qquad (44)$$

where $$|\epsilon| \leq \frac{292\,\theta^9 f_m{}^8}{105\cdot 9!} + \frac{3!\theta^{11} f_m{}^{10}}{4\cdot 11!},$$

or

$$|\epsilon| \leq \frac{\theta^9 f_m{}^8}{130487} + \frac{\theta^{11} f_m{}^{10}}{26611200}.$$

8. **General Case of Symmetrical Points. — Balanced Systems.** Let the expression

$$\theta^{(i+1)}[\{K^i_{-n}\}f^i_{-n_i}+\{K^i_{-(n-1)}\}f^i_{-(n-1)_i}+\cdots+\{K^i_{(n-1)}\}f^i_{(n-1)_i} + \{K^i_n\}f^i_{n_i}] \qquad (45)$$

be known as an *element* and denoted by $\{E^i\}$, where the subscripts refer to the points

$$\{c^i_{-n}\}\theta,\ \{c^i_{-(n-1)}\}\theta,\ \cdots,\ \{c^i_{(n-1)}\}\theta,\ \{c^i_n\}\theta, \qquad (46)$$

which are subject *only* to the condition that

$$0 \leq -\{c^i_{-j}\} = \{c^i_j\} \quad \text{where } j = 1, 2, 3, \cdots n_i, \qquad (47)$$

and i is an integer.

Thus except for symmetry the points are chosen arbitrarily. The $\{c^i_j\}$'s and the $\{K^i_j\}$'s are constants, the upper subscripts within the braces merely identifying the element. Also, since there is no interdependence between the points of different elements, we must refer to the points of $\{E^i\}$ by the subscripts

$$1_i,\ 2_i,\ 3_i,\ \cdots n_i;$$

however, where there is no ambiguity superfluous symbols will be omitted; thus we write $\{K^i_j\}$ and $\{c^i_j\}$ instead of $\{K^i_{j_i}\}$ and $\{c^i_{j_i}\}$. Also, by definition $f^k_{j_i} = \dfrac{d^k}{d\theta^k} f(\{c^i_j\}\theta)$ as in previous sections. (48)

We now arbitrarily construct[15] a certain number of elements and denote $\Sigma\{E^i\}$ by S. $\phi(\theta)$ is defined by the relation

$$\phi(\theta)=\int_{-\theta}^{\theta}f(x)dx-S=\int_{-\theta}^{\theta}f(x)dx-\Sigma\{E^i\}, \tag{49}$$

S being temporarily restricted to have no mid-terms. Let θ^{i+1} be taken within the bracket of $\{E^i\}$ so that

$$\{E^i\}=\sum_{j=-n}^{j=n}\theta^{i+1}\{K_j^i\}f_j^i \qquad (j\pm 0).$$

We now take successive derivatives of $\phi(\theta)$, *no attempt being made to collect terms*. Terms not of the form $\theta\alpha$ are called *principal terms;* also, we define[16]

$$\{E_r^i\}=\frac{d^r}{d\theta^r}\{E^i\}, \tag{50}$$

and

$$\{E_r\}=\frac{d^r}{d\theta^r}\int_{-\theta}^{\theta}f(x)dx.$$

Since we are later going to place $\theta=0$, we need consider only the principal terms of the various derivatives of $\phi(\theta)$. The *expression for the principal terms of* $\{E_r^i\}$ can be found directly from Leibnitz's rule, which is

$$(uv)^r=u^rv+\binom{r}{1}u^{r-1}v^1+\binom{r}{2}u^{r-2}v^2+\cdots$$
$$\cdots+\binom{r}{2}u^2v^{r-2}+\binom{r}{1}u^1v^{r-1}+uv^r \tag{51}$$

where the upper subscripts refer to derivatives; and

$$\binom{n}{m}=\frac{n(n-1)(n-2)\cdots(n-m+1)}{m!}=\binom{n}{n-m} \quad \text{if } n>m>0,$$
$$\binom{n}{m}=0 \quad \text{if } n<m, \tag{52}$$
$$\binom{n}{0}=\binom{n}{n}=1.$$

[15] The number and choice of elements is arbitrary, as is the number and choice of the points of the different elements. The symmetry of the points is the principal restriction, the other restriction being one which will be discussed later — that the system for determining the K's be determinate.

[16] It is evident that in the above notation $f^0(x)=f(x)$, $\{E^0\}=\{E^0{}_0\}=\theta\Sigma\{K\}f$; also that $\{E_0\}=\int_{-\theta}^{\theta}f(x)dx=\{E\}$.

Thus we can write[17] $(uv)^r = \sum_{i=0}^{i=r} \binom{r}{i} u^{r-i} v^i.$ (53)

We now place $\{K_j^i\}\,\theta^{i+1} = u,$
$f_j^i = f^i(\{c_j^i\}\,\theta) = v;$

and note that $u^k = 0$ if $k > (i+1),$
$u^k = \theta\alpha$ if $k < (i+1);$ (54)

hence, since we are interested only in the principal term

$$k = (i+1) \quad \text{and} \quad u^{(i+1)} = \{K_j^i\}\,\overline{i+1}\,!.$$

Thus the expression for the principal term of $\{E_r^i\}$ due to the term $\theta^{i+1}\{K_j^i\}\,f_j^i$ of $\{E^i\}$ is

$$\binom{r}{i+1}\{K_j^i\}\,\overline{i+1}\,!\,\{c_j^i\}^{r-i-1} f^{r-1}(\{c_j^i\}\,\theta). \tag{55}$$

For convenience, we shall place (55) in a somewhat different form. Let us define:

$$\{n, m\} = n\,(n-1)(n-2)\cdots(n-m) \quad \text{if } n \geq (m+1),$$
$$\{n, m\} = 0 \quad \text{if } n < (m+1)$$

where $m \geq 0$; hence

$$\{n, m\} = \overline{m+1}!\binom{n}{m+1},$$
$$\{n, n-1\} = \{n, (n-2)\} = n!.$$

Thus (55) is reduced to the following expression:

$$\{\mathbf{K}_j^i\}\,\{\mathbf{r}, \mathbf{i}\}\,\{\mathbf{c}_j^i\}^{r-i-1} f^{r-i}(\{\mathbf{c}_j^i\}\,\theta) \tag{56}$$

where

$$\begin{matrix}\{\mathbf{r}, \mathbf{i}\} = \mathbf{r}(\mathbf{r}-\mathbf{1})(\mathbf{r}-\mathbf{2})\cdots(\mathbf{r}-\mathbf{i}) & \text{if } \mathbf{r} \geq (\mathbf{i}+\mathbf{1}) \\ \{\mathbf{r}, \mathbf{i}\} = \mathbf{0} & \text{if } \mathbf{r} < (\mathbf{i}+\mathbf{1})\end{matrix} \quad \mathbf{i} \geq \mathbf{0}.$$

Also, we have from (50)

$$\{E_r\} = f^{r-1}(\theta) + (-1)^{r-1} f^{r-1}(-\theta). \tag{57}$$

The K's are to be determined so that for any f

$$0 = \phi^1(0) = \phi^2(0) = \phi^3(0) = \cdots; \tag{58}$$

[17] *Pierpont: Theory of Functions of Real Variables*, v. 1, p. 253; also p. 61.

therefore, as in (39), we place $\theta=0$. The θa terms fall out, and only principal terms remain; also, when $\theta=0$, $f_{j_i}^{r-1}=f^{r-1}(\{c_j^i\}0)$ $=f^{r-1}(0)$ for any j_i. From (56) and (57) we see that the principal terms of $\phi^r(\theta)$ involve f^{r-1}, and *no other derivative*[18] *of* f; thus we can factor $f^{r-1}(0)$ out of $\phi^r(0)$, as was done in the case of (5), and thus obtain

$$\phi^r(0)=0=Q_r f^{r-1}(0) \tag{59}$$

where Q_r involves the K's and c's. Since $\phi^r(0)$ is to vanish for any f, we have

$$Q_r=0, \qquad r=1, 2, 3, \cdots, \tag{60}$$

as the system for determining the K's.

Let $\{Q_r^i\}$ denote the terms of Q_r due to $\{E^i\}$, and let $\{Q_r\}$ denote those due to $\int_{-\theta}^{\theta} f(x)dx$, or $\{E\}$. These expressions will be known as *elements*, since

$$Q_r=\{Q_r\}+\Sigma\{Q_r^i\}. \tag{61}$$

From (57), (56) and (47) we have respectively:

$$\{Q_r\}=1+(-1)^{r-1}, \tag{62}$$

$$-\{Q_r^i\}=\sum_{j=-n}^{j=-1}\{K_j^i\}\{r, i\}\{c_j^i\}^{r-i-1}+\sum_{j=1}^{j=n}\{K_j^i\}\{r, i\}\{c_j^i\}^{r-i-1}, \tag{63}$$

$$-\{Q_r^i\}=\sum_{j=1}^{j=n}\{r, i\}\{c_j^i\}^{r-i-1}[\{K_j^i\}+(-1)^{r-i-1}\{K_{-j}^i\}]. \tag{64}$$

An element is said to be *balanced* if all of its terms are of the same sign; otherwise it is *unbalanced*.[19] From (62) and (64) it is evident that $\{Q_r\}$ is balanced if r is odd, and that $\{Q_r^i\}$ is balanced if $(r-i)$ is odd. Q_r contains an even number of unknowns, for such is true of each $\{Q_r^i\}$; the system (60) will therefore generally consist of an even number of equations. Half of these

[18] The upper subscript of f is $(r-1)$, which is independent of i.

[19] From (63) it is seen that $\{Q_r^i\}$ consists of two symmetrical sums, each of which contains terms of like sign; though the two sums may be of different sign. Thus, schematically

$[+ + \cdots + - \cdots - -]$ = unbalanced element

$[- - \cdot - - \cdot - -]$ = balanced element.

equations will be satisfied, and half of the unknowns will drop out if

$$\begin{aligned} \left\{\mathbf{K}_{-j}^{i}\right\} &= \left\{\mathbf{K}_{j}^{i}\right\}, \qquad i \text{ even}, \\ -\left\{\mathbf{K}_{-j}^{i}\right\} &= \left\{\mathbf{K}_{j}^{i}\right\}, \qquad i \text{ odd}, \end{aligned} \tag{65}$$

which substitution renders all elements of a given Q_r in the same state of balance or unbalance. (60) thus reduces to

$$Q_r = 0, \qquad r = 1, 3, 5, \cdots,$$

which, from (61), is seen to be

$$\sum^{S} (r, i) \sum_{j=1_i}^{j=n_i} \left\{C_j^i\right\}^{r-i-1} \left\{K_j^i\right\} = 1$$

where $r = 1, 3, 5, \cdots$.

Let us now suppose that $\left\{E^i\right\}$ has a mid-term, $\theta^{i+1}\left\{K_0^i\right\} f^i(0)$. Since $f^i(0)$ is a constant, this mid-term contributes but one principal term to (60), which term falls in $\left\{Q_{i+1}^i\right\}$. Consequently, since we do not wish to disturb the condition that $Q_{\text{even}} = 0$, we restrict $(i+1)$ to odd values; therefore $\left\{E^i\right\}$ *may have a mid-term only if* i *is even*. As an example, $\left\{E^2\right\}$ of (37) consists only of the mid-term. The terms "balanced" and "unbalanced" have been applied to elements having even numbers of terms; however, we shall extend the definition of a "balanced element" so as to include elements having odd numbers of terms. *The term "unbalanced" is confined to elements of even numbers of terms.*

We now *place a limit upon* $|\epsilon|$ by considering $\phi^{\gamma}(\theta)$, which we assume to have no mid-terms, or terms of the form $\left\{K_0^i\right\} \overline{i+1}!\, f^i(0)$. Here

$$0 < \gamma \text{ even} \leq (2\Sigma_1 + \Sigma_2) \tag{66}$$

where $\Sigma_1 =$ total number of mid-terms of S,
$\Sigma_2 =$ total number of other terms.

The $\theta\alpha$ terms must now be considered, this being done by the aid of Leibnitz's rule. Referring to (51) and (54), we see that terms involving the same order of derivative of f may be grouped together into elements of $\phi^{\gamma}(\theta)$; also that the elements involving odd derivatives of f are unbalanced, whereas the remaining elements are balanced.[20] Noting (47) we now use the law of the mean

[20] The extension of the meaning of the terms "balanced" and "unbalanced" is obvious [19]

to eliminate odd derivatives of f, *having first collected terms.* Proceeding as in (42) we place

$$f_m^{\mu} = \text{greatest } |f^{\mu}(x)| \text{ in the interval bounded by the two outermost points.} \tag{67}$$

Again collecting terms, and integrating successively[21] noting (58), we have

$$|\epsilon| \leq \frac{\theta^{\gamma+1} f_m^{\gamma} \sigma_1}{\overline{\gamma+1}!} + \frac{3!\theta^{\gamma+3} f_m^{\gamma+2} \sigma_2}{\overline{\gamma+3}!} + \frac{5!\theta^{\gamma+5} f_m^{\gamma+4} \sigma_3}{\overline{\gamma+5}!} + \cdots; \tag{68}$$

where there are T terms altogether, and the σ's are constants which can be computed, since the K's have been determined. We see from (51) and (54) that there are $(i+1)$ terms[22] of the form "$\theta\alpha$" in $\{E_\gamma^i\}$; consequently, if we denote the largest i found among the terms of S other than mid-terms by I, we have

$$T = \frac{I}{2} + 1 \quad \text{if } I \text{ is even,}$$

$$T = \frac{I+1}{2} + 1 \quad \text{if } I \text{ is odd.} \tag{69}$$

Another limit can be placed upon $|\epsilon|$ *by considering* $\phi^{\gamma+1}(\theta)$, where γ is subject to the condition (66), and $\phi^{\gamma+1}(\theta)$ has no middle terms, or terms of the form $\{K_0^i\}\,\overline{i+1}!\,f^i(0)$. As before we see that elements of $\phi^{\gamma+1}(\theta)$ which involve odd derivatives of f are unbalanced, while the others are balanced; consequently we collect terms, use the law of the mean to eliminate odd derivatives of f, collect absolute values, and integrate successively. The result is an expression of the form (68); however, in this case

$$T = \frac{I}{2} + 2 \quad \text{if } I \text{ is even,}$$

$$T = \frac{I+1}{2} + 1 \quad \text{if } I \text{ is odd.} \tag{70}$$

[21] We have used f_m^{γ} in order to take particular derivatives of f outside of the integral sign. Obviously this can be accomplished by means of expressions of the form $f_1^{\gamma}(\xi)$, where ξ lies somewhere in the interval bounded by the two outermost points; in fact the symbols f_m^{γ} and $f^{\gamma}(\xi)$ can be interchanged in deriving expressions limiting $|\epsilon|$. With due regard for signs, an expression for ϵ can be derived which contains but a finite number of terms.

[22] This is true except in the case where $\gamma < i$, in which no difficulty is encountered, and of which no further mention will be made.

Comparing (69) and (70) we see that if I is even, the two methods lead to limits of different form; whereas if I is odd, the forms of the limits derived by the two methods are identical. Let the first term of (68) be known as the *principal term* of the error, and let the *order* of the error refer to Ω, the upper limit of γ. Thus from (66) we have

$$\Omega = 2\Sigma_1 + \Sigma_2 . \tag{71}$$

As an example, by inspection of the formula

$$\int_{-\theta}^{\theta} f(x)dx = \theta[A_{-2}f_{-2} + A_2 f_2] + \theta^2[B_{-2}f_{-2}{}^1 + B_{-1}f_{-1}{}^1 + B_1 f_1{}^1 + B_2 f_2{}^1] + \theta^3 C_0 f_0{}^2 + \epsilon,$$

we see that $I = 1$; $\therefore \quad T = 2$.

Also $\Omega = 2+6 = 8$, $\quad A_{-2} = A_2$, $\quad -B_{-2} = B_2$, $\quad -B_{-1} = B_1$;

hence $|\epsilon| \leq \dfrac{\theta^9 f_m{}^8 \sigma_1}{9!} + \dfrac{3!\theta^{11} f_m{}^{10} \sigma_2}{11!}$,

as was seen to be the case in (44).

As another example, consider the Cotes formulas for $(2n+1)$ points. Here $\Omega = (2n+2)$,

$$I = 0; \qquad \therefore \quad \begin{matrix} T = 1 & \text{for (69),} \\ T = 2 & \text{for (70).} \end{matrix}$$

We have already noted that if I is odd, (69) and (70) give limits of the same form. We shall now investigate the relative values of σ_1 obtained by the two methods, which methods will be denoted by A and B respectively. $(\sigma_1)_A$ depends upon the terms of $\phi^\gamma(\theta)$ which represent the first two terms of Leibnitz's rule (51), since the law of the mean is applied to the principal terms. On the other hand $(\sigma_1)_B$ is dependent only upon the principal terms of $\phi^{\gamma+1}(\theta)$. Writing out one pair of symmetrical terms which affect σ_1, and denoting the other terms by R_A and R_B respectively, we have, using (55) and (51),

$$-\phi^\gamma(\theta) = 2\{c_j^i\}\,\theta[\{K_j^i\}\,\overline{i+1}!\,\{c_j^i\}^{\gamma-i-1} f^\gamma(\xi)]\binom{\gamma}{i+1}$$

$$+2\theta[\{K_j^i\}\,\overline{i+1}!\,\{c_j^i\}^{\gamma-i} f^\gamma(\{c_j^i\}\,\theta)]\binom{\gamma}{i} + R_A, \tag{72}$$

$$-\phi^{\gamma+1}(\theta) = 2[\{K_j^i\}\,\overline{i+1}!\,\{c_j^i\}^{\gamma-1} f^\gamma(\{c_j^i\}\,\theta)]\binom{\gamma+1}{i+1} + R_B,$$

where $$-\{c_j^i\}\theta<\xi<\{c_j^i\}\theta$$

and the 2's are due to the fact that the law of the mean has been applied and terms collected in the case of this one pair of terms. Integrating $\phi^{\gamma+1}(\theta)$ between 0 and θ we have, noting (67) and (58),

$$F=|\phi^\gamma(\theta)|_A\leq\left|\{2\theta\{K_j^i\}\overline{i+1}!\{c_j^i\}^{\gamma-i}f_m^\gamma\}\binom{\gamma}{i+1}\right.$$
$$\left.+\{2\theta\{K_j^i\}\overline{i+1}!\{c_j^i\}^{\gamma-i}f_m^\gamma\}\binom{\gamma}{i}\right|+R_A \quad (73)$$

$$G=|\phi^\gamma(\theta)|_B\leq\left|\{2\theta\{K_j^i\}\overline{i+1}!\{c_j^i\}^{\gamma-i}f_m^\gamma\}\binom{\gamma+1}{i+1}\right|+R_B;$$

and since $$\binom{\gamma}{i+1}+\binom{\gamma}{i}=\binom{\gamma+1}{i+1},$$

we have $F=G$ if the R's are neglected. Since (73) applies to any set of symmetrical points, we conclude that in the *case where all points are distinct* from each other, and from $(-\theta)$ and θ (the points of $\{E\}$ being included with the others), the result is

$$(\sigma_1)_A=(\sigma_1)_B, \quad (74)$$

where the external subscripts merely refer to the method of derivation. This result applies directly to the formulas of Gauss and Tchebychef.

We now consider the case where the points are not distinct; that is, different elements may now have points in common. Let us denote the first term of F by F_1, the second by F_2, and the first term of G by G_1. Collecting terms at the point j we have

$$F_j=|\Sigma_j F_1|+|\Sigma_j F_2|$$
$$G_j=|\Sigma_j G_1|=|\Sigma_j(F_1+F_2)|\leq F_j,$$

for the terms are collected before the law of the mean is applied to obtain F_1. Neglecting terms which do not affect σ_1,

$F=\Sigma F_j$
$G=\Sigma G_j$ (Sums of positive terms taken over all points);
$\therefore F\geq G.$

From the above we thus conclude that

$$(\sigma_1)_A=(\sigma_1)_B \quad (75)$$

if and only if each $\Sigma_j F_1$ *and* $\Sigma_j F_2$ *are of the same sign, which need not be the same for different points; otherwise*

$$(\sigma_1)_A > (\sigma_1)_B .$$

We see from (73) that F_1 and F_2 are of the same sign; thus in applying (75) we may *neglect distinct pairs of points;* also, it is obvious that $\{E\}$ is not distinguished from the other elements.

We now consider the other σ's in the case where all points are distinct, though σ_1 is the most important; and write one pair of terms which affect $\sigma_{\frac{p+2}{2}}$, denoting all other terms by R_A and R_B respectively,

$$F \leq \{2\{c_j^i\}\theta^{p+1}|\{K_j^i\}|\overline{i+1}!\{c_j^i\}^{p+\gamma-i-1}f_m^{\gamma+p}\}\frac{\binom{\gamma}{i+1-p}}{p!}$$

$$+\{2\theta^{p+1}|\{K_j^i\}|\overline{i+1}!\{c_j^i\}^{p+\gamma-i}f_m^{\gamma+p}\}\frac{\binom{\gamma}{i-p}}{\overline{p+1}!}+R_A; \tag{76}$$

$$G \leq \{2\{c_j^i\}\theta^{p+1}|\{K_j^i\}|\overline{i+1}!\{c_j^i\}^{p+\gamma-i-1}f_m^{\gamma+p}\}\frac{\binom{\gamma+1}{i+2-p}}{(p+1)\overline{p-1}!}$$

$$+\{2\theta^{p+1}|\{K_j^i\}|\overline{i+1}!\{c_j^i\}^{p+\gamma-i}f_m^{\gamma+p}\}\frac{\binom{\gamma+1}{i+1-p}}{\overline{p+1}!}+R_B;$$

where p is even and $\qquad p \geq (i+1-\gamma).$ (77)

Here the terms have been collected, the law of the mean applied, and $|\phi^{\gamma+1}(\theta)|$ integrated once. $p=1, 2, 3, \cdots$, starting with the second term of (51); or in other words p numbers the $\theta\alpha$ terms from the lowest to the highest power of θ. Let the four explicit terms of (76) be denoted by F_1, F_2, G_1, and G_2 respectively. From (77) we have

$$\frac{1}{\gamma+1} \leq \frac{1}{i+2-p}, \tag{78}$$

which is algebraically the following:

$$\frac{1}{p}+\frac{i+1-p}{(\gamma-i+p)p(p+1)}\leq\frac{\gamma+1}{(i+2-p)(p+1)}+\frac{\gamma+1}{(\gamma-i+p)p(p+1)}. \tag{79}$$

Multiplying both sides of (79) by

$$\left\{2\theta^{p+1}\right|\left\{K_j^i\right\}\left|\overline{i+1}\,!\left\{c_j^i\right\}^{p+\gamma-i}f_m^{\gamma+p}\right\}\frac{\binom{\gamma}{i+1-p}}{\overline{p-1}\,!}$$

we see that (78) becomes

$$\left\{F_1+F_2\right\}\leq\left\{G_1+G_2\right\},$$

which shows that *if all points are distinct*

$$(\sigma)_A\leq(\sigma)_B \tag{80}$$

for any σ except σ_1, which was discussed separately.[13] Further analysis of the higher σ's will be omitted, since no simple rigorous result is thereby reached.

As an example let us consider Simpson's rule for one interval. Applying (75) we have

$$F_1=2\left(1-\frac{4}{3}\right)\beta<0;\quad\text{where }\beta=+\text{ factor},$$

$$F_2=-2\left(\frac{1}{3}\right)\beta<0,\text{ and therefore is of the same sign as }F_1;$$

$$\therefore(\sigma_1)_A=(\sigma_1)_B,$$

which is verified by the following numerical results[12]:

$$|\epsilon|_A\leq\frac{\theta^5 f_m{}^4}{90}\text{ and }|\epsilon|_B\leq\frac{\theta^5 f_m{}^4}{90}+\frac{\theta^7 f_m{}^6}{3780}.$$

On the other hand, in the case of (14)

$$F_1=2\left(1-\frac{6\cdot7}{45}\right)\beta_1>0,$$

$$F_2=-\frac{2\cdot7}{45}\beta_1<0;$$

$$\therefore(\sigma_1)_A>(\sigma_1)_B,$$

[13] The last σ must be discussed separately, the procedure being of the same nature as that in the case of σ_1. The result agrees with (80), I being odd. If I is even, the last $(\sigma)_B$ has no corresponding $(\sigma)_A$; hence (80) is still satisfied.

which is verified by the result

$$|\epsilon|_A \leq \frac{\theta^7 f_m^6}{8400} \text{ and } |\epsilon|_B \leq \frac{\theta^7 f_m^6}{15120} + \frac{19\theta^9 f_m^8}{30 \cdot 9!}.$$

We have to this point used methods A and B to obtain expressions which limit $|\epsilon|$; however, attention will now be given to a method whereby *the error can be limited by a series, finite or infinite*, which involves increasing powers of θ and increasing orders of the derivatives of $f(x)$. Let us consider an element of $\phi'(\theta)$, the derivative taken as the starting point in deriving the limit upon $|\epsilon|$; and let us expand each of the terms of this element into a power series *about the same point* using Taylor's formula, finite or infinite. We then operate upon this new expression in the usual manner, obtaining a limit[24] of $|\epsilon|$. Obviously, any or all of the terms of any or all of the elements may be treated in this manner, the particular Taylor's series used in each case being arbitrary, as is the point about which the expansion is made. The mid-point is, however, usually the most convenient; also we see that certain σ's, σ_1 in particular, may be reduced in this manner to the least values possible. This is because the addition that is later performed is algebraic instead of absolute. If Taylor's infinite series is used upon every term of $\phi'(\theta)$ and the usual procedure is then followed, the limit obtained for $|\epsilon|$ becomes an infinite series; thus we have an expression[8] *which converges uniformly to* ϵ, *or* $\phi(\theta)$. Thus in the case of Simpson's rule,[12] we have

$$\epsilon = -\frac{\theta^5 f_0^4}{90} - \frac{\theta^7 f_0^6}{1890} - \frac{\theta^9 f_0^8}{90720} - \cdots. \tag{81}$$

[24] In certain cases, the error can be reduced considerably by a suitable choice of the co-ordinate system. For example, in the case of the semi-circle Cotes' formula for three points gives a poor approximation of $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$, but an exact result in the case of $\int_0^{\frac{\pi}{2}} \frac{(2a\cos\theta)^2}{2}\, d\theta$; although both integrals are expressions for the area of the semi-circle.

9. **Special Cases of Balanced Systems. — Euler's Formula.** We now apply the theory developed in Section 8, and consider the case where

$$i=0, 1, 3, 5, \cdots \text{ odd, and } -\{c_{-1}^{i}\}=\{c_{1}^{i}\}=1; \text{ hence}$$

$$S=\theta[\{K_{-1}^{0}\}f_{-1}+\{K_{1}^{0}\}f_{1}]+\theta^{2}[\{K_{-1}^{1}\}f_{-1}^{1}+\{K_{1}^{1}\}f_{1}^{1}]$$
$$+\theta^{4}[\{K_{-1}^{3}\}f_{-1}^{3}+\{K_{1}^{3}\}f_{1}^{3}]+\cdots, \tag{82}$$

from which it follows, due to (65), that

$$\{K_{-1}^{0}\}=\{K_{1}^{0}\}$$
$$-\{K_{-1}^{i}\}=\{K_{1}^{i}\} \text{ for } i\pm 0.$$

The following system is due to (60), (56), and (57),

$$\begin{aligned}
1&=\{K_{1}^{0}\}\{1, 0\}+\{K_{1}^{1}\}\{1, 1\}+\{K_{1}^{3}\}\{1, 3\}+\cdots\\
1&=\{K_{1}^{0}\}\{3, 0\}+\{K_{1}^{1}\}\{3, 1\}+\{K_{1}^{3}\}\{3, 3\}+\cdots\\
1&=\{K_{1}^{0}\}\{5, 0\}+\{K_{1}^{1}\}\{5, 1\}+\{K_{1}^{3}\}\{5, 3\}+\cdots\\
1&=\{K_{1}^{0}\}\{7, 0\}+\{K_{1}^{1}\}\{7, 1\}+\{K_{1}^{3}\}\{7, 3\}+\cdots\\
&\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots
\end{aligned} \tag{83}$$

Noting that $\{n, m\}=0$ if $n<(m+1)$, we see that all terms of (83) that lie above the principal diagonal are 0. This system is determinate, for the determinant of the first k equations is the product of the terms of the principal diagonal:

$$\Delta_{k}=1!\cdot 3!\cdot 5!\cdot 7!\cdots\overline{2k-1}!\neq 0.$$

Also
$$\begin{aligned}
I&=(2k-3),\\
T&=k,\\
\Sigma_{1}&=0,\\
\Omega=\Sigma_{2}&=2k.\\
&S \text{ contains } k \text{ elements.}
\end{aligned} \tag{84}$$

We shall now derive a special formula based upon the fact, indicated by (71), that Ω is more affected by mid-terms than by others. In this case the formula will consist entirely of mid-terms; hence, noting the condition placed upon such terms in Section 8, we have

$$S=\theta\{K_{0}^{0}\}f_{0}+\theta^{3}\{K_{0}^{2}\}f_{0}^{2}+\theta^{5}\{K_{0}^{4}\}f_{0}^{4}+\cdots\theta^{2k-1}\{K_{0}^{2k-2}\}f_{0}^{2k-2}.$$

The system for determining the K's is

$$\begin{aligned}
2&=\{K_0^0\}\\
2&=\quad 3!\{K_0^2\}\\
2&=\qquad\quad 5!\{K_0^4\}\\
&\cdots\\
2&=\qquad\qquad\quad \overline{2k-1}!\{K_0^{2k-2}\}.
\end{aligned}$$

We have also

$$T=1,$$
$$\Sigma_2=0,$$
$$\Omega=2\Sigma_1=2k.$$

S contains k elements.

The result is therefore

$$\int_{-\theta}^{\theta} f(x)dx=2\theta\left\{f_0+\frac{\theta^2 f_0^2}{3!}+\frac{\theta^4 f_0^4}{5!}+\cdots\frac{\theta^{2k-2}f_0^{2k-2}}{\overline{2k-1}!}\right\}+\epsilon$$

where
$$|\epsilon|\leq\frac{2\theta^{2k+1}f_m^{2k}}{\overline{2k+1}!}. \tag{85}$$

This formula appears to be the same as that obtained by integrating Taylor's expansion about the mid-point. As an example let us use (85) to approximate $\frac{1}{2}\int_{-\pi}^{x}\cos x\,dx=\sin x$.

Here $\theta=x, f_0=1, f_0^2=-1, f_0^4=1, f_0^6=-1, \cdots$.

Thus
$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\frac{(-1)^{k-1}x^{2k-1}}{\overline{2k-1}!}+\epsilon$$

where
$$|\epsilon|\leq\frac{2x^{2k+}}{\overline{2k+1}!}.$$

The formulas of Cotes, Gauss, Tchebychef, and Christoffel are other examples of balanced systems.

10. **Possibility of Leaving c's as Unknown Constants.** In Section 8 it was assumed that the c's were chosen arbitrarily and then fixed. This was done merely to fix ideas, for it is obvious that the theory developed is independent of the type of unknowns. We thus see that certain K's and c's may be fixed, subject to con-

ditions (47) and (65); and the remaining ones determined by placing derivatives of $\phi(\theta)$ equal to zero in the usual manner, the method being that of Sections 4, 5 and 6. In general

$$\Omega = 2M \tag{86}$$

where M is the number of unknowns remaining after the application of (47) and (65).

As an example, let $S = \theta\{K^0_0\}f_0 + \theta^2[-f^1_{-1} + f^1_1]$,

where $\{K^0_0\}$, $\{c^1_{-1}\}$, and $\{c^1_1\}$ are to be determined.

We have immediately from (56), and (69):

$$\{K^0_0\} = 2, \quad -\{c^1_{-1}\} = \{c^1_1\} = \frac{1}{6}, \quad I = 1, \quad T = 2, \quad \Omega = 4;$$

hence $$\int_{-\theta}^{\theta} f(x)dx = 2\theta f_0 + \theta^2\left\{-f^1\left(-\frac{\theta}{6}\right) + f^1\left(\frac{\theta}{6}\right)\right\} + \epsilon$$

where $$|\epsilon| \leq \frac{59\theta^5 f^4_m}{27 \cdot 5!} + \frac{11\theta^7 f^6_m}{1944 \cdot 7!},$$ using method B.

11. **Determinateness of Results. — Arbitrary Systems.** The theory developed in Sections 8, 9 and 10 was developed entirely upon the basis of practical convenience, since the formulas that are of practical importance are special cases of it. The arbitrary nature of the theory is shown by the restriction of mid-terms to $\{E^{\text{even}}\}$.

There is obviously no difficulty encountered in the general case where the number and choice of the elements of S are arbitary, the number and choice of the terms of the various elements are without restriction, certain K's and c's are fixed arbitrarily, and those remaining are determined by placing successive derivatives of $\phi(\theta)$ equal to zero. A limit upon $|\epsilon|$ may be found by handling any of these derivatives in the usual manner. The one restriction upon S is that the system which results for determining the unknowns is determinate. The case where there is but one element in S was fully considered in Section 3; however, the general case is much more difficult.

We shall consider here only the cases where the c's have been fixed and only K's are to be determined, other cases being left to trial.

Let us suppose that though the number of elements is arbitrary, no element is omitted; thus

$$i = 0, 1, 2, 3, \cdots I. \tag{87}$$

The system is determinate if and only if

$$\Delta \neq 0$$

where Δ is the determinant of the system, which after eliminating mid-terms by developing by minors is of the following form:

$$\Delta = B \begin{vmatrix} a_{10} & \cdots a_{10} & 0 & \cdots 0 & 0 & \cdots 0 & \cdots \\ a_{11}c_1 & \cdots a_{11}c_2 & a_{30} & \cdots a_{30} & 0 & \cdots 0 & \cdots \\ a_{12}c_1^2 & \cdots a_{12}c_2^2 & a_{31}c_3 & \cdots a_{31}c_4 & a_{50} & \cdots a_{50} & \cdots \\ a_{13}c_1^3 & \cdots a_{13}c_2^3 & a_{32}c_3^2 & \cdots a_{32}c_4^2 & a_{51}c_5 & \cdots a_{51}c_6 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1c_1^q & \cdots a_1c_2^q & a_3c_3^{q-1} & \cdots a_3c_4^{q-1} & a_5c_5^{q-2} & \cdots a_5c_6^{q-2} & \cdots \end{vmatrix} \tag{88}$$

where the a's and c's are constants representing the $\{r, i\}$'s and $\{c_j^i\}$'s of (56) respectively, and B is not zero. It is seen that each element contributes a block of terms to this determinant. We now develop this determinant by minors with respect to the first column, and so obtain a polynomial $P_1(c_1)$ that is of degree d_1 where $d_1 \leq q$. The c's of the first block are obviously roots of $P_1(c_1)$; however, as opposed to the case of Section 3, *all* of the roots are not known. As it is known that they exist we write, using the factor theorem,

$$\frac{\Delta}{B} = P_1(c_1) = (c_1 - \alpha_{11})(c_1 - \alpha_{12}) \cdots (c_1 - \alpha_{1,d_1})\Delta_1 = \Pi_1\Delta_1 \tag{89}$$

where Δ_1 is the cofactor of $a_{1,d_1}c_1^{d_1}$, and it is seen that Δ_1 and the α_1's do not contain c_1. Since Δ_1 is a determinant of the same type as Δ, we may write

$$\Delta_1 = P_2(c_2) = (c_2 - \alpha_{21})(c_2 - \alpha_{22}) \cdots (c_2 - \alpha_{2,d_2})\Delta_2 = \Pi_2\Delta_2,$$

the procedure being that of (89), where Δ_2 and the α_2's contain neither c_1 nor c_2. Proceeding in this way we finally have

$$\Delta = B\Pi_1 \cdot \Pi_2 \cdot \Pi_3 \cdots \Pi_\omega \cdot V \tag{90}$$

where development with respect to the upper row is made when the last column of each block is encountered; hence V is a Vandermonde determinant, which is not zero since the c's of each block are necessarily distinct. The α's of Π_ω are determined by the c's of V; hence once these c's are fixed we have a *discrete set*

of values which if assigned to the c of Π_n *will result in the vanishing of* Δ. Moreover, only these values of c will cause Δ to vanish. We now fix this c and immediately have a discrete set of values which cannot be assigned to the c of $\Pi_{n-1}\cdots$, and so on. We thus see that Δ *can be, but generally is not zero;* also that if Δ is zero in a particular case, an adjustment of the c's is all that is required to render the system determinate. It is therefore evident that S *need not be altered in form.*

Let us now suppose that k successive elements are omitted, there being elements on both sides of the gap so formed; then the block on the right has $(k+1)$ more rows of zeroes than has the adjacent block on the left. In deriving (90), successive columns of (88) were taken out, the degree of the determinant being lowered; however, we now continue this process only to the point where there are $(k+1)$ columns to the left of the gap. Laplace's development then throws the determinant into the product of two others of the type (88). If there are several gaps, the process is continued only to the point where it is first possible to use Laplace's development, the resulting pair of determinants being considered separately as before. Thus we finally have either a product of gapless determinants each of the type (88), or the conclusion that Δ is zero.

A system is usually determinate if the first p equations involve no fewer than p unknowns. As an example, every S must have an element $\{E^0\}$.

As a further example, there can be no formula of the form

$$\int_{-\theta}^{\theta} f(x)dx = \theta[\{K^0_{-1}\}f(-\theta)+\{K^0_{1}\}f(\theta)]+\theta^2\{K^1_{0}\}f^1(0)+\epsilon,$$

$$\text{for } \Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 2 & 2 \\ 3 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0.$$

12. **Uniqueness of Results.** Let us suppose that by the above method there has been obtained an expression S, which is a special case of (36); and that the system which determines the unknowns of S consists of t equations, and is uniquely determinate. Also let us suppose that in some other way there has been derived a formula, Σ, which gives exact results when applied to a poly-

nomial of degree $\leq \delta$, where δ is a given number; and that Σ is similar in form to S, by which we mean that the same unknowns were determined in both cases. It will now be shown that if

$$\delta \geq (t-1), \tag{91}$$

then $$S \equiv \Sigma.$$

To this end we place

$$\phi(\theta) = \int^{\theta} f(x)dx - \sum,$$

and differentiate successively. Applying the law of the mean, $f^{\gamma}(c\theta) = f^{\gamma}(0) + c\theta f^{\gamma+1}(\xi)$, to refer each principal term of $\phi^{\gamma+1}(\theta)$ to the mid-point, we have

$$\phi^{\gamma+1}(\theta) = Q_{\gamma+1} f^{\gamma}(0) + \left\{ \begin{array}{c} \text{Terms involving higher} \\ \text{derivatives of } f \end{array} \right\} \tag{92}$$

where $Q_{\gamma+1}$ involves only the unknowns, which were determined by the other method. We now integrate (92) successively between the limits 0 and θ, using the law of the mean upon the higher order terms to keep mean values of the derivatives of f outside of the integral sign. The result is:

$$\phi(\theta) = \frac{Q_{\gamma+1}\theta^{\gamma+1} f^{\gamma}(0)}{\gamma+1!} + \lambda_{\gamma+1} \tag{93}$$

where $\lambda_{\gamma+1}$ represents terms involving higher derivatives of f. By hypothesis we know that if

$$f(x) = P_{\gamma}(x) = \textit{Any polynomial of degree } \gamma \tag{94}$$

with the condition that $\gamma \leq \delta$, we have from (93) that

$$\textit{Error} = \phi(\theta) = 0 = \frac{Q_{\gamma+1}\theta^{\gamma+1} P_{\gamma}{}^{\gamma}(0)}{\gamma+1!}; \tag{95}$$

since $\lambda_{\gamma+1}$ vanishes due to the order of the derivatives which it involves. Applying (94) and noting that the interval of integration is arbitrary, we fix an interval and choose $P_{\gamma}(x)$ so that $P_{\gamma}{}^{\gamma}(0) \neq 0$; hence (95) becomes

$$Q_{\gamma+1} = 0 \quad \text{if } \gamma \leq \delta. \tag{96}$$

Placing $\gamma=0, 1, 2, 3, \cdots \delta$, we obtain a system of $(\delta+1)$ equations that are satisfied by the unknowns of Σ. But the unknowns of S were *determined* by the relations:

$$Q_{\gamma+1}=0 \qquad \text{where } \gamma=0, 1, 2, 3, \cdots (t-1);$$

therefore

$$\mathbf{S} \equiv \Sigma \tag{97}$$

if and only if $\delta \geq (t-1)$.

We now consider some applications of (97) noting that the formulas to which we shall refer have been derived as indicated in Section 1, and which therefore give exact results when applied to polynomials of a degree smaller than some given number. Denoting the total number of points by N, we have the following cases:

a. Formulas of Cotes: $\delta=N-1, t=N; \therefore \delta=t-1$.

b. Formulas of Tchebychef: $\delta=N, t=N+1; \therefore \delta=t-1$.

c. Formulas of Gauss: $\delta=2N-1, t=2N; \therefore \delta=t-1$.

d. Formulas of Euler: $\delta=N-1, t=N; \therefore \delta=t-1$.

Since (97) is satisfied in each of these cases, the titles of Sections 2, 4, 5, and 9 are justified. We now consider two more cases:

e. Formula of Simpson: $\delta=3, t=[N \text{ odd}]>1; \therefore \delta \geq t-1$ if and only if $N=3$; which merely states that unless $N=3$, Simpson's formula differs from the corresponding one of Cotes.

f. Integration of Taylor's Series. Referring to (85) we have: $\delta=2k-2, t=2k-1; \therefore \delta=t-1$, and (85) is identical with the integral of Taylor's expansion about the mid-point.

All methods of procedure do not lead to the same limit upon $|\epsilon|$ even though all may be of the same form; however, if we take $\phi^{r}(\theta)$ as the starting point, expand all of its terms in Taylor's *infinite* series about a common point, and then proceed in the usual manner, the result is an *expansion of* ϵ *in powers of* θ, which is necessarily unique and coincident with the corresponding one of Taylor. Therefore this expansion and those of a similar nature which have been derived by other methods[5,7] are identical.

A NEW METHOD IN TAUBERIAN THEOREMS

By Norbert Wiener

The classical theorem of Abel states that if $\Sigma a_n = A$, then

$$\lim_{x \to 1-0} \Sigma a_n x^n = A. \tag{1}$$

The converse of this theorem is not unconditionally true.[1] Tauber[2] showed that if (1) is true and $\lim na_n = 0$, it follows that $\Sigma a_n = A$. Hardy and Littlewood[3] have shown that (1) and the auxiliary condition that $na_n > -K$ form a sufficient hypothesis for this conclusion. Theorems of this type, which involve some hypothesis on the rapidity of growth of the terms of a series, and under this assumption proceed from the given summability of the series by some particular method to its convergence in the ordinary sense, they term Tauberian theorems.

The methods of Hardy and Littlewood are difficult, and involve a repeated differentiation of arbitrarily high order. They are furthermore but little systematized, so that each particular theorem involves a considerable amount of manipulation peculiar to itself. To avoid this, Robert Schmidt[4] introduced the machinery of the theory of moments, and at the cost of a considerable amount of complication in the case of the theorems already proved, gave a general theory of Tauberian theorems adequate to deal with the majority of known cases.

There is however one particular Tauberian theorem which has proved refractory even to the improved tools of the Schmidt method. Hardy and Littlewood have shown[5] that the classical theorem of the distribution of primes, to the effect that the

[1] Cf. $a_n = (-1)^n$.

[2] A. Tauber, Ein Satz aus der Theorie der unendlichen Reihen. *Monatsch. f. Math. u. Phys.* 8 (1897), pp. 273–277.

[3] Hardy and Littlewood, Tauberian Theorems concerning Power Series and Dirichlet's Series whose Coefficients are Positive. *Proc. Lond. Math. Soc.*, Series 2, 13 (1913), pp. 174–191.

[4] R. Schmidt, Über divergente Folgen und lineare Mittelbildungen. *Math. Ztschr.* 22 (1925), pp. 89–152.

[5] Hardy-Littlewood, New Proof of the Prime Number Theorem, *Quarterly Journal of Pure and Applied Math.*, 46 (1915), pp. 215–219.

number of primes less than n is asymptotically represented by $n/\log n$, is an elementary consequence of the Tauberian theorem to the effect that if $na_n > -K$ and

$$\lim_{w \to 0} \Sigma\, a_n \frac{nw\, e^{-nw}}{1-e^{-nw}} = A, \tag{2}$$

then $\Sigma a_n = A$. This theorem they are only able to prove by a recourse to the prime number theorem itself.

The present paper was suggested to the author by the investigations of Mr. Schmidt, to whom the author is also indebted to the suggestion to apply his method to the Tauberian theorem with the hypothesis (2). It bears the same relation to the Schmidt theory which the theory of the Fourier integral bears to the method of moments and the theory of the Laplace transformation. In other words, it involves the theory of complex moments. It is the author's opinion that it brings not merely a systematization, but a simplification, of a large region in the theory of Tauberian theorems. It is in particular adequate for a direct treatment of the case with hypothesis (2).

Hypotheses (1) and (2) are alike of the form

$$\lim_{w \to 0} \Sigma\, a_n N(nw) = A, \tag{3}$$

where $N(0) = 1$. If we put $a(x) = \sum_{k=0}^{[x]} a_k$,

(3)

$$\lim_{w \to 0} \int_0^\infty N(nw)\, da(n) = A. \tag{4}$$

If N vanishes sufficiently rapidly at infinity, we may integrate by parts and (4) becomes

$$\lim_{w \to 0} \int_0^\infty a(u/w)\, N'(u)\, du = -A. \tag{5}$$

On this hypothesis, together with a further hypothesis restricting the rate of growth of a, we wish to prove that

$$\lim_{u \to \infty} a(u) = A. \tag{6}$$

It will be seen that expression (5) is a mean of the values of $a(u)$ for large values of u. Our Tauberian theorems may then be put in the form: if a function has a mean value of a certain type for large values of its arguments, and if the rate of growth of the function is subject to certain restrictions, then the function has a limiting value at infinity. Closely related to theorems of this type are theorems which read as follows: if a function has a mean of a certain given type at infinity, and its growth is restricted in a certain given manner, then it has a mean of a certain other given type at infinity. We shall find it more convenient to prove a theorem of this latter type, and then to deduce from this theorems of a more obviously Tauberian nature.

Our key theorem reads as follows: *Let* $M_1(x)$ *and* $M_2(x)$ *be two real functions which are* $O\left(\frac{1}{x(\log x)^2}\right)$ *both at* 0 *and at* ∞, *and which are bounded over every interval* $(\epsilon, 1/\epsilon)$. *Let* $M_1(x)$ *be non-negative and measurable, and let*

$$\int_0^\infty M_1(x)x^{iu}dx \neq 0. \quad [u = \text{any real number}]$$

Let $M_2(x)$ *be continuous except for a finite number of finite jumps. Let* $\phi(x)$ *be a non-negative measurable function such that*

(a) $$\lim_{y\to 0}\int_0^\infty \phi\left(\frac{x}{y}\right)M_1(x)dx = A\int_0^\infty M_1(x)dx;$$

(b) $$\int_0^\infty \phi\left(\frac{x}{y}\right)M_1(x)dx \text{ is bounded.} \quad [0<y<\infty]$$

Then

(c) $$\lim_{y\to 0}\int_0^\infty \phi\left(\frac{x}{y}\right)M_2(x)dx = A\int_0^\infty M_2(x)dx.$$

In order to prove this theorem, we shall introduce the transformations

$$x = e^\xi;\ y = e^\eta;\ xM_1(x) = N_1(\xi);\ xM_2(x) = N_2(\xi);\ \phi(x) = f(\xi).$$

Our theorem now assumes the form: *let* $N_1(\xi)$ *and* $N_2(\xi)$ *be two*

bounded functions which are $O(1/\xi^2)$ *at* $\pm\infty$. *Let* $N_1(\xi)$ *be measurable and non-negative, and let*

$$\int_{-\infty}^{\infty} N_1(\xi) e^{iu\xi}\, d\xi \neq 0. \quad [-\infty < u < \infty]$$

let $N_2(\xi)$ *be continuous, except for a finite number of finite jumps. Let* $f(\xi)$ *be a non-negative measurable function such that*

(*a*) $\lim_{\eta \to -\infty} \int_{-\infty}^{\infty} f(\xi-\eta)\, N_1(\xi) d\xi = A \int_{-\infty}^{\infty} N_1(\xi) d\xi;$

(*b*) $\int_{-\infty}^{\infty} f(\xi-\eta)\, N_1(\xi) d\xi$ *is bounded.* $[-\infty < \eta < \infty]$.

Then

(*c*) $\lim_{\eta \to -\infty} \int_{-\infty}^{\infty} f(\xi-\eta)\, N_2(\xi) d\xi = A \int_{-\infty}^{\infty} N_2(\xi) d\xi.$

In order to prove the theorem in this form, let Φ be the class of all bounded measurable functions $N_2(\xi)$ for which (*c*) holds whenever $f(\xi)$ is a non-negative function satisfying (*a*) and (*b*). If $R(\xi)$ is an absolutely integrable function,

$$S(\xi) = \int_{-\infty}^{\infty} N_1(\xi-\zeta) R(\zeta) d\zeta$$

belongs to Φ. For we have

$$\int_{-\infty}^{\infty} f(\xi-\eta) S(\xi) d\xi = \int_{-\infty}^{\infty} f(\xi-\eta) d\xi \int_{-\infty}^{\infty} N_1(\xi-\zeta) R(\zeta) d\zeta$$

$$= \int_{-\infty}^{\infty} R(\zeta) d\zeta \int_{-\infty}^{\infty} f(\xi-\eta+\zeta)\, N_1(\xi) d\xi,$$

as the order of integration may be inverted owing to the absolute integrability of R and the non-negative character of f and N_1. Let us put

$$\int_{-\infty}^{\infty} f(\xi-\eta)\, N_1(\xi) d\xi - A \int_{-\infty}^{\infty} N_1(\xi) d\xi = \psi(\eta).$$

The function $\psi(\eta)$ is then a bounded function approaching 0 as η tends toward $-\infty$. We have

$$\left|\int_{-\infty}^{\infty} R(\zeta)\psi(\eta-\zeta)d\zeta\right| \leq \left|\int_{-\infty}^{B} R(\zeta)\psi(\eta-\zeta)d\zeta\right| + \left|\int_{B}^{\infty} R(\zeta)\psi(\eta-\zeta)d\zeta\right|$$

$$\leq \max|\psi(\eta)|\int_{-\infty}^{B}|R(\zeta)|d\zeta$$

$$+ \max_{\zeta \geq B}|\psi(\eta-\zeta)|\int_{-\infty}^{\infty}|R(\zeta)|d\zeta.$$

It follows at once that

$$\overline{\lim_{\eta\to-\infty}}\left|\int_{-\infty}^{\infty} R(\zeta)\psi(\eta-\zeta)d\zeta\right| \leq \max|\psi(\eta)|\int_{-\infty}^{B}|R(\zeta)|d\zeta,$$

and as B is arbitrary, that

$$\overline{\lim_{\eta\to-\infty}}\left|\int_{-\infty}^{B} R(\zeta)\psi(\eta-\zeta)d\zeta\right| = 0,$$

or as we may write it, that

$$\overline{\lim_{\eta\to-\infty}}\int_{-\infty}^{\infty} f(\xi-\eta)S(\xi)d\xi = A\int_{-\infty}^{\infty} R(\zeta)d\zeta\int_{-\infty}^{\infty} N_1(\xi)d\xi$$

$$= A\int_{-\infty}^{\infty} S(\xi)d\xi.$$

The boundedness of S follows from that of N_1 and the absolute integrability of R. Hence S belongs to Φ.

Let us put

$$\nu_1(u) = \int_{-\infty}^{\infty} N_1(\xi)e^{iu\xi}d\xi.$$

By the Plancherel theory of the Fourier transform,

$$\mu(u) = \operatorname*{limit\ in\ mean}_{B\to\infty}\int_{-B}^{B} i\xi N_1(\xi)e^{iu\xi}d\xi$$

exists as a measurable function the square of whose modulus is Lebesgue integrable from $-\infty$ to ∞.[6] We have

[6] M. Plancherel, Contribution à l'étude de la représentation d'une function arbitraire par des intégrales définies. *Rendiconti di Palermo*, 30 (1910), pp. 289–335.

$$\int_{\alpha}^{\beta}\mu(u)du = \int_{-\infty}^{\infty} i\xi N_1(\xi)\left[\int_{\alpha}^{\beta} e^{iu\xi}du\right]d\xi$$

$$= \nu_1(\beta) - \nu_1(\alpha).$$

The function $\nu_1(u)$ is thus a non-vanishing function which is an indefinite integral.

Let $S(\xi)$ be a bounded function which is $O(1/\xi^2)$ at $\pm\infty$. Let

$$\sigma(u)=\int_{-\infty}^{\infty}S(\xi)e^{iu\xi}d\xi,$$

and let

$$\tau(u)=\text{l. i. m.}_{B\to\infty}\int_{-B}^{B}i\xi S(\xi)e^{iu\xi}d\xi.$$

As above, we have

$$\int_{-\alpha}^{\beta}\tau(u)du=\sigma(\beta)-\sigma(\alpha).$$

Let us make the further restriction that

$$\tau(u)=0 \qquad [|u|\geq D].$$

I then say that $S(\xi)$ belongs to Φ.

The proof is as follows: consider

$$\omega(u)=\frac{\nu_1(u)\tau(u)-\mu(u)\sigma(u)}{[\nu_1(u)]^2}$$

It is easily seen that $\sigma(u)$ as well as $\tau(u)$ must vanish for $|u|\geq D$, as it is constant outside this range and is quadratically summable. Over the range $(-D, D)$, $\nu_1(u)$ and $\sigma(u)$ are bounded continuous functions, and $\nu_1(u)$ does not vanish. Hence $\omega(u)$ is a bounded measurable function vanishing outside $(-D, D)$, such that the square of its absolute value is Lebesgue integrable. We can take $D=\pi$ in what follows without any essential restriction of the argument.

Let $\nu_1(u)$ have the Fourier series $\sum_{-\infty}^{\infty}a_ne^{inu}$. Then $\mu(u)$ will have the Fourier series $\Sigma' in\, a_ne^{inu}$. Similarly, let $\sigma(u)$ have the Fourier series Σb_ne^{inu}, which means that $\tau(u)$ will have the Fourier series $\Sigma\, in\, b_ne^{inu}$. Let

$$\nu_1{}^N(u)=\sum_{-N}^{N}a_ne^{inu}\left(1-\frac{|n|}{N}\right);$$

$$\mu^N(u)=\sum_{-N}^{N}in\,a_ne^{inu}\left(1-\frac{|n|}{N}\right);$$

$$\sigma^N(u)=\sum_{-N}^{N} b_n e^{inu}\left(1-\frac{|n|}{N}\right);$$

$$\tau^N(u)=\sum_{-N}^{N} in\, b_n e^{inu}\left(1-\frac{|n|}{N}\right).$$

It is clear that

$$\int_{-\pi}^{\pi}\left|\omega(u)-\frac{\nu^N{}_1(u)\tau^N(u)-\mu^N(u)\sigma^N(u)}{[\nu^N{}_1(u)]^2}\right|^2 du$$

$$=\int_{-\pi}^{\pi}\left|\frac{\nu_1(u)\tau(u)[\nu^N{}_1(u)]^2-\mu(u)\sigma(u)[\nu^N{}_1(u)]^2-\nu^N{}_1(u)\tau^N(u)[\nu_1(u)]^2}{[\nu_1(u)]^2[\nu^N{}_1(u)]^2}\right|du$$

The functions $\nu^N{}_1(u)$, $\sigma^N(u)$ and $1/\nu^N{}_1(u)$ are all uniformly bounded in u and N and converge uniformly to the corresponding functions without the N. It can then be shown by a simple but tedious argument that

$$\underset{N\to\infty}{\text{l.i.m.}}\frac{\nu^N{}_1(u)\tau^N(u)-\mu^N(u)\sigma^N(u)}{[\nu^N{}_1(u)]^2}=\omega(u).$$

Similarly,

$$\lim_{N\to\infty}\frac{\sigma^N(u)}{\nu^N{}_1(u)}=\frac{\sigma(u)}{\nu_1(u)}$$

uniformly in u. Since

$$\int_{\alpha}^{\beta}\frac{\nu^N{}_1(u)\tau^N(u)-\mu^N(u)\sigma^N(u)}{[\nu^N{}_1(u)]^2}du=\frac{\sigma^N(\beta)}{\nu^N{}_1(\beta)}-\frac{\sigma^N(\alpha)}{\nu^N{}_1(\alpha)}$$

we have

$$\int_{\alpha}^{\beta}\omega(u)du=\frac{\sigma(\beta)}{\nu_1(\beta)}-\frac{\sigma(\alpha)}{\nu_1(\alpha)}.$$

Let us put

$$R(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sigma(u)}{\nu_1(u)}e^{-iu\xi}\,du. \tag{7}$$

Then

$$S(\xi)=\int_{-\infty}^{\infty}N_1(\xi-\zeta)R(\zeta)d\zeta.$$

We also have

$$R(\xi)=\frac{1}{2\pi i\xi}\int_{-\infty}^{\infty}\omega(u)e^{-iu\xi}du.$$

Since $\int_{-\infty}^{\infty}\omega(u)e^{-iu\xi}du$ is quadratically summable, it follows by a simple use of the Schwarz inequality that

$$\int_{B}^{\infty}|R(\xi)|d\xi \text{ and} \int_{-\infty}^{-B}|R(\xi)|d\xi$$

are bounded. Combining this fact with the boundedness of R, which we deduce from (7), we see that $R(\xi)$ is absolutely integrable from $-\infty$ to ∞. It follows from our previous lemma that $S(\xi)$ belongs to Φ.

If we already know certain members of the class Φ, we may enlarge the class in the following manner. Let $S(\xi)$ be a function such that, if any positive ϵ is given, we can find two members of Φ, S_1 and S_2, such that $S_1(\xi)\leq S(\xi)\leq S_2(\xi)$, while

$$\int_{-\infty}^{\infty}[S_2(\xi)-S_1(\xi)]d\xi<\epsilon.$$

Then S itself belongs to Φ. For

$$\overline{\lim_{\eta\to-\infty}}\int_{-\infty}^{\infty}f(\xi-\eta)S(\xi)d\xi\leq\lim_{\eta\to-\infty}\int_{-\infty}^{\infty}f(\xi-\eta)S_2(\xi)d\xi$$

$$\leq A\left[\int_{-\infty}^{\infty}S(\xi)d\xi+\epsilon\right];$$

$$\underline{\lim_{\eta\to-\infty}}\int_{-\infty}^{\infty}f(\xi-\eta)S(\xi)d\xi\geq A\left[\int_{-\infty}^{\infty}S(\xi)d\xi-\epsilon\right].$$

Since ϵ is arbitrarily small

$$\lim_{\eta\to-}\int_{-\infty}^{\infty}f(\xi-\eta)S(\xi)d\xi=A\int_{-\infty}^{\infty}S(\xi)d\xi,$$

and S belongs to Φ.

As an example of a function S which can be proved in this manner to belong to Φ, let us take

$$S(\xi)=\frac{p}{(\xi-q)^2+r^2}.$$

We shall carry the proof through in the case $S(\xi)=1/(1+\xi^2)$; the method is general.

We have

$$S(\xi)=\frac{1}{2}\int_{-\infty}^{\infty}e^{-|u|}e^{i\xi u}du.$$

Let us consider the slightly modified function.

$$T(\xi)=\frac{1}{2}\int_{-B}^{B}[e^{-|u|}-e^{-B}]e^{i\xi u}du$$

which belongs to Φ as a result of our previous discussion. We have

$$\begin{aligned}T(\xi)&=\frac{1}{2\xi i}\int_{-B}^{B}[e^{-|u|}-e^{-B}]de^{i\xi u}\\&=\frac{1}{2\xi i}\int_{0}^{B}\cdot e^{i\xi u}e^{-u}du-\frac{1}{2\xi i}\int_{-B}^{0}e^{i\xi u}e^{u}du\\&=R\left(\frac{1}{\xi i}\int^{B}e^{u(i\xi-1)}du\right)=R\left\{\frac{1}{\xi i(\xi i-1)}[e^{B(i\xi-1)}-1]\right\}\\&=\frac{1-e^{-B}\cos B\xi}{\xi^2+1}-\frac{e^{-B}\sin B\xi}{\xi(\xi^2+1)}.\end{aligned}$$

Hence

$$|T(\xi)-S(\xi)|<(1+B)e^{-B}S(\xi),$$

from which it follows that for large enough values of B,

$$[1+(1+B)e^{-B}]^{-1}T(\xi)<S(\xi)<[1-(1-B)e^{-B}]^{-1}T(\xi).$$

If we make use of the fact that

$$\lim_{B\to\infty}(1+B)e^{-B}=0,$$

we see that we may put

$$[1+(1+B)e^{-B}]^{-1}T(\xi)=S_1(\xi);\ [1-(1+B)e^{-B}]^{-1}T(\xi)=S_2(\xi),$$

and apply our last lemma. It follows at once that $S(\xi)$ belongs to Φ.

Another function which belongs to Φ is

$$Q_A(\xi)=2/\pi\int_0^A\left(1-\frac{u}{A}\right)\frac{\sin u\cos u\xi}{u}\,du.$$

We have

$$Q_A(\xi)=G(A(\xi+1))-G(A(\xi-1)),$$

where

$$G(\xi)=\frac{1}{\pi}\int_0^1\frac{(1-v)\sin v\,\xi}{v}\,dv.$$

Now,

$$G(0)=0;\quad G'(\xi)=\frac{1}{\pi}\int_0(1-v)\cos v\xi\,dv$$

$$=\frac{1-\cos\xi}{\pi\xi^2}.$$

Hence

$$G(\xi)=\int_0^\xi\frac{1-\cos z}{\pi z^2}\,dz$$

$$=\int_0^\infty\frac{1-\cos z}{\pi z^2}\,dz-\int_\xi^\infty\frac{dz}{\pi z^2}+\int_\xi^\infty\frac{\cos z}{\pi z^2}\,dz$$

$$=\frac{1}{2}-\frac{1}{\pi\xi}+O(1/\xi^2),$$

as may be seen by considering the graph of $\cos z/\pi z^2$. It follows that

$$Q_A(\xi)=\frac{1}{\pi A(\xi-1)}-\frac{1}{\pi A(\xi+1)}+O(1/A^2\xi^2)$$

$$=O(1/A\xi^2)$$

as $A\xi^2$ becomes positively infinite in any manner whatever, pro-

vided ξ remains larger than some constant greater than 1 (or less than some constant less than -1). Hence the function

$$Q_A(\xi) - \frac{1}{B(\xi^2+1)}$$

is negative over $(1+\eta, \infty)$ and $(-\infty, -1-\eta)$ for large enough values of A.

As A increases, $Q_A(\xi)$ tends uniformly to

$$\operatorname{sgn}(\xi+1) - \operatorname{sgn}(\xi-1)$$

except in the neighborhood of ± 1, while it always lies between 0 and 1. Furthermore,

$$\int_{-\infty}^{\infty} Q_A(\xi)\,d\xi = 2.$$

It follows that if we put $S(\xi) = \operatorname{sgn}(\xi+1) - \operatorname{sgn}(\xi-1)$, we may put

$$(1+\eta)Q_A(\xi(1-\eta)) = S_2(\xi);$$

$$(1-\eta)Q_A(\xi(1+\eta)) - \frac{1}{B(\xi^2+1)} = S_1(\xi);$$

for $Q_A(K\xi)$ contains only a limited range of frequencies, and so belongs to Φ, $\dfrac{1}{B(\xi^2+1)}$ belongs to Φ, the difference between two members of Φ belongs to Φ, and we can so choose η, B and A that

$$(1+\eta)Q_A(\xi(1-\eta)) > \operatorname{sgn}(\xi+1) - \operatorname{sgn}(\xi-1)$$

$$> (1-\eta)Q_A(\xi(1+\eta)) - \frac{1}{B(\xi^2+1)}, \quad (8)$$

while

$$\int_{-\infty}^{\infty} \left[(1+\eta)Q_A(\xi(1-\eta)) - (1-\eta)Q_A(\xi(1+\eta)) + \frac{1}{B(\xi^2+1)} \right] d\xi \quad (9)$$

$$= 2\left[\frac{1+\eta}{1-\eta} - \frac{1-\eta}{1+\eta} \right] + \frac{\pi}{B} < \epsilon.$$

To make this determination, we first choose η and B so that (9)

holds, and then determine A in such a manner as to make (8) hold. It follows that sgn $(\xi+1)-$sgn $(\xi-1)$ belongs to Φ.

The same method of proof will show that any step function of a finite number of steps, vanishing for large positive and negative arguments, belongs to Φ, for the sum of a number of members of Φ belongs to Φ, and the argument in the case of any function of a single step proceeds exactly as in the case of sgn $(\xi+1)-$ sgn$(\xi-1)$. It follows that any function continuous except for a finite number of finite jumps, vanishing for large positive and negative arguments also belongs to Φ, for we can determine step-functions S_1 and S_2 for it.

Now let $N_2(\xi)$ be any bounded function which is $O(1/\xi^2)$ at $\pm\infty$, which is continuous except for a finite number of finite jumps. Let

$$|N_2(\xi)|<P/\xi^2+1$$

for all ξ. We put

$$S_1(\xi)=N_2(\xi); \qquad [|\xi|<M]$$
$$S_1(\xi)=-P/\xi^2+1; [|\xi|\geq M].$$
$$S_2(\xi)=N_2(\xi); \qquad [|\xi|<M]$$
$$S_2(\xi)=P/\xi^2+1. \quad [|\xi|\geq M]$$

It is immediately clear that $N_2(\xi)$ belongs to Φ, as S_1 and S_2 are the sums of finite sets of functions in Φ. Our main theorem is thus proved.

It is worth while noting that our main theorem has at least a partial inverse which is valid. We shall suppose that $\int_0^\infty M_1(x)dx\neq 0$.

If $\int_0^\infty M_1(x)x^{i\alpha}dx=0$, then let us put

$$\phi(x)=x^{i\alpha}+x^{-i\alpha}.$$

Then

(a) $$\lim_{y\to 0}\int_0^\infty \phi\left(\frac{x}{y}\right)M_1(x)dx=A\int_0^\infty M_1(x)dx; \quad [A=0]$$

(b) $$\int_0^\infty \phi\left(\frac{x}{y}\right)M_2(x)dx \text{ is bounded.}$$

Unless $M_2(x)$ satisfies the special condition

$$\int_0^\infty M_2^{(x)} x^{iu} dx = 0,$$

we shall *not* have

$$(c) \quad \lim_{y \to 0} \int_0^\infty \phi\left(\frac{x}{y}\right) M_2(x) dx = A \int_0^\infty M_2(x) dx.$$

It is only the restriction on the order of magnitude and the boundedness of M_1 which prevents us from regarding this as a perfect inverse of the fundamental theorem. Except for this, our theorem is a "best possible theorem," and no alternative method such as the method of moments can go substantially beyond it.

Our main theorem is only implicitly Tauberian. We wish to make the transition to the theorems of a more obviously Tauberian character. To this end, we put

$$M_2(x) = \begin{cases} 1; & [x < T] \\ 0; & [x \geq T] \end{cases}; \tag{10}$$

$$M_1(x) = -\int_x^\infty \frac{F'(y)}{y} dy.$$

$M_2(x)$ satisfies all the conditions necessary for the hypothesis of our theorem. If $F'(y)$ is bounded, and is $O(y^{-2-\delta})$ at infinity, we shall have

$$M_1(x) = O(\log x) \text{ at } 0;$$

$$M_1(x) = O(x^{-3-\delta}) \text{ at } \infty;$$

and *a fortiori*

$$M_1(x) = O\left(\frac{1}{x(\log x)^3}\right)$$

at 0 and ∞. The condition

$$\int_x^\infty M_1(x) x^{iu} dx \neq 0$$

may be written

$$H(u) = \int_0^\infty x^{iu} dx \int_x^\infty \frac{F'(y)}{y} dy \neq 0 \qquad (u \text{ real}).$$

$H(u)$ is an analytic function of u in the strip $0 > I(u) \geq -1$, and

is continuous over the region $0 \geq I(u) \geq -1$. In the strip of analyticity,

$$H(u) = \int_0^\infty \frac{F'(y)}{y}\, dy \int_0^y x^{iu} dx$$

$$= \frac{1}{iu+1} \int_0^\infty F'(y) y^{iu} dy$$

$$= -\frac{iu}{iu+1} \int_0^\infty F(y) y^{iu-1} dy.$$

If then the function $\int_0^\infty F(y)\, y^{z-1} dy$ is analytic in the neighborhood of the line $R(z)=0$, with the exception of a pole of order 1 at $z=0$, and does not vanish on this line,

$$\int_0^\infty M_1(x) x^{iu} dx \neq 0.$$

The only remaining assumption concerning M_1 alone is that it is non-negative. This will be satisfied if $F'(x)$ is non-positive.

Examples of functions F satisfying our conditions are

$$F(x) = e^{-x};$$

$$F(x) = \frac{xe^{-x}}{1-e^{-x}}$$

In the first case,

$$\int_0^\infty F(y) y^{z-1} dy = \int_0^\infty e^{-y} y^{z-1} dy = \Gamma(z);$$

and in the second,

$$\int_0^\infty F(y) y^{z-1} dy = \int_0^\infty \frac{e^{-y}}{1-e^{-y}}\, y^z dy = \zeta(z+1)\, \Gamma(z+1).$$

For such functions F, if $\phi(x)$ is non-negative (or simply bounded below, as we may easily generalize), and

(a) $$-\lim_{\epsilon \to 0} \int_0^\infty \phi\left(\frac{x}{\epsilon}\right) \int_x^\infty \frac{F'(y)}{y}\, dy = AF(0);$$

(*b*) $$\int_0^\infty \phi\left(\frac{x}{\epsilon}\right)\int_x^\infty \frac{F'(y)}{y}dy \text{ is bounded,}$$

we have by our theorem

(*c*) $$\lim_{E\to\infty}\frac{1}{E}\int_0^E \phi(x)dx = A.$$

We may however write

$$-\int_0^\infty \phi\left(\frac{x}{\epsilon}\right)dx\int_x^\infty \frac{F'(y)}{y}dy = -\int_0^\infty \frac{F'(y)}{y}dy\int_0^y \phi\left(\frac{x}{\epsilon}\right)dx$$

$$= -\int_0^\infty F'(y)dy\,\frac{\epsilon}{y}\int_0^{y/\epsilon}\phi(\xi)d\xi.$$

Let us now introduce the supposition that we may express $\frac{1}{y}\int_0^y \phi(x)dx$ in the form

$$\frac{1}{y}\int_0^y \phi(x)dx = \int_0^y \psi(x)dx. \tag{11}$$

From the lower boundedness of $\phi(x)$ and the assumptions

(*a'*) $$AF(0) = \lim_{\epsilon\to 0} -\int_0^\infty F'(y)dy\int_0^{y/\epsilon}\psi(x)dx$$

(*b'*) $$\int_0^\infty F'(y)dy\int_0^{y/\epsilon}\psi(x)dx \text{ is bounded}$$

we can deduce by our theorem

(*c'*) $$\int_0^\infty \psi(x)dx = A.$$

Let $\psi(x)$ be any function summable over every finite interval and vanishing for arguments $\leq\eta$ and satisfying the relation

$$\psi(x) > -K/x.$$

Let us put

$$\phi(x) = x\psi(x) + \int_0^x \psi(\xi)d\xi.$$

We then see that (11) is valid. Proposition (a') takes the form,

$$AF(0) = -\lim_{\epsilon \to 0} \int_0^\infty F'(y)dy \int_0^{y/\epsilon} \psi(x)dx$$

$$= \lim_{\epsilon \to 0} \lim_{B \to \infty} -\int_0^B F'(y)dy \int_0^{y/\epsilon} \psi(x)dx$$

$$= \lim_{\epsilon \to 0} \lim_{B \to \infty} \int_0^{B/\epsilon} \left\{ F(\epsilon x) - F(B) \right\} \psi(x)dx.$$

We shall show that we can deduce this from the assumption that

$$(a'') \qquad AF(0) = \lim_{\epsilon \to 0} \int_0^\infty F(\epsilon x)\psi(x)dx$$

together with the auxiliary assumption that $F(x)$ is of the order of magnitude of some negative power of x at infinity, and the assumptions which we have already made concerning F and ψ. Under these assumptions,

$$\int_\eta^\infty F(\epsilon x)\left[\psi(x) + \frac{K}{x}\right]dx = O(1) + K\int_{\eta\epsilon}^\infty F(\xi)\frac{d\xi}{\xi}$$

$$= O(\log \epsilon).$$

Since the expression $\psi(x) + \dfrac{K}{x}$ is non-negative, we can replace $F(x)$ by any smaller expression, and in particular by an expression equal to a constant from $x = 0$ to $x = L > 0$. Then

$$\int_0^{1/\epsilon} \left[\psi(x) + \frac{K}{x}\right]dx = O(\log \epsilon).$$

A repetition of our argument will give us

$$\int_0^{1/\epsilon} \psi(x)dx = O(\log \epsilon).$$

Hence

$$\lim_{B \to \infty} F(B) \int_0^{B/\epsilon} \psi(x)dx = 0,$$

and (a') is established as a consequence of (a'').

In order to prove proposition (b') as a consequence of (a''), let us notice that

$$\left|\int_0^\infty F'(y)dy\int_0^{y/\epsilon}\psi(x)dx\right| \leq C_1\left|\int_{\eta\epsilon}^\infty F'(y)\,(\log y-\log\epsilon)dy\right|$$
$$\leq C_1\left|-F(\eta\epsilon)(\log\eta\epsilon-\log\epsilon)-\int_{\eta\epsilon}^\infty\frac{F(y)}{y}dy\right|$$
$$\leq C_2$$

for all values of ϵ greater than a given value, C_1 and C_2 being properly chosen constants. In combination with (a'), this establishes (b').

The expression $x\psi(x)$ is bounded below. If then $\int_0^x\psi(\xi)d\xi$ is bounded below, (c') is valid whenever (a') is.

We wish to prove that under suitable hypotheses, the boundedness of $\int_0^x\psi(\xi)d\xi$ follows from that of $x\psi(x)$. We shall assume that $F(0)=1$. We define $M_2(x)$ as in (10). Then

$$\int_\epsilon^\infty[F(\epsilon x)-M_2(\epsilon x)]\psi(x)dx \leq \int_0^{T/\epsilon}\frac{K}{x}\,(1-F(\epsilon x))dx$$
$$+\mu\int_0^\infty\psi(x)[F(\epsilon x)-F(\lambda\epsilon x)]dx$$
$$+\mu\int_0^{T/\epsilon}\frac{K}{x}\,[F(\epsilon x)-F(\lambda\epsilon x)]dx$$
$$+\int_{T/\epsilon}^\infty\frac{K}{x}[(\mu-1)F(\epsilon x)-\mu F(\lambda\epsilon x)]dx,$$

provided for a $\mu>1$ and $\lambda>1$

$$F(x)\geq\frac{\mu}{\mu-1}\,F(\lambda x)\qquad[x>T]$$

Under this condition,

$$\int_0^\infty[F(\epsilon x)-M_2(\epsilon x)]\psi(x)dx$$
$$\leq K\left[\int_0^T\frac{1-F(x)}{x}dx+\mu\int^T\frac{F(x)-F(\lambda x)}{x}dx\right.$$
$$\left.+\int_T^\infty\frac{(\mu-1)F(x)-\mu F(\lambda x)}{x}\,dx\right]+\mu\int_0^\infty[F(\epsilon x)-F(\lambda\epsilon x)]\psi(x)dx.$$

It then follows from (a') and (b') that

$$\int_0^\infty M_2(\epsilon x)\psi(x)dx = \int_0^{1/\epsilon} \psi(x)dx$$

is bounded below.

To gather these results into our second fundamental theorem: *Let $F(x)$ be a differentiable function assuming the value 1 at 0 and tending to 0 at infinity. Let $F'(x)$ be nowhere positive, bounded, $O(x^{-2-\delta})$ at ∞, and let $F(x)$ be subject to the relation*

$$F(\lambda x) < \theta_1 F(x), \quad [x > T].$$

where $\lambda > 1$, $0 < \theta_1 < 1$. Let

$$\int_0^\infty F(y)y^{s-1}dy$$

be analytic in the neighborhood of the line $R(z) = 0$ with the exception of a pole of order 1 at $z = 0$, and let it have no zeros on this line. Let $\psi(x)$ be any measurable function (the restriction that it shall not vanish in the neighborhood of the origin is obviously unnecessary) satisfying the relation

$$\psi(x) > -K/x.$$

Then if

$$\lim_{\epsilon \to 0} \int_0^8 F(\epsilon x)\psi(x)dx = A$$

it follows that

$$\int_0^\infty \psi(x)dx = A.$$

Possible values of F are

$$F(x) = e^{-x};$$

$$F(x) = \frac{xe^{-x}}{1-e^{-x}}.$$

In particular, let

$$\psi(x) = a_n \qquad [n-1 < x \leq n]$$

where

$$a_n > -K/n.$$

Let us investigate the meaning of the condition

$$\lim_{\epsilon \to 0} \int_0^\infty F(\epsilon x)\psi(x)dx = A.$$

We have

$$\int^\infty F(\epsilon x)\psi(x)dx = \sum_1^\infty a_n \int_{n-1}^n F(\epsilon x)dx$$

$$= \sum_1^\infty a_n F(\epsilon n) + \sum_1^\infty \left(a_n + \frac{K}{n}\right)\left[\int_{n-1}^n F(\epsilon x)dx - F(\epsilon n)\right]$$

$$+ \sum_1^\infty \frac{K}{n}\left[F(\epsilon n) - \int_{n-1}^n F(\epsilon x)dx\right].$$

By the law of the mean,

$$\int_{n-1}^n F(\epsilon x)dx - F(\epsilon n) < -\epsilon F'(\epsilon(n-\theta_n)). \quad [0 \leq \theta_n \leq 1].$$

Let us introduce the new hypothesis,

$$-F'(x) < CF(x+1) < De^{-\Delta x}. \tag{12}$$

It follows that

$$\left|\sum_1^\infty \frac{K}{n}\left[F(\epsilon n) - \int_{n-1}^n F(\epsilon x)dx\right]\right| \leq \sum_1^\infty \frac{K\epsilon De^{-\Delta(n-1)\epsilon}}{n}$$

$$= K\epsilon De^{\Delta\epsilon}\log\,(1-e^{-\Delta\epsilon});$$

and that

$$\sum_1^\infty \left(a_n + \frac{K}{n}\right)\left[\int_{n-1}^n F(\epsilon x)dx - F(\epsilon n)\right] \leq C\epsilon \sum_1^\infty \left(a_n + \frac{K}{n}\right)F(\epsilon n)$$

$$\leq C\epsilon \sum_1^\infty a_n F(\epsilon n) + C\epsilon KD \sum_1^\infty \frac{e^{-\Delta(\epsilon n-1)}}{n}$$

$$= C\epsilon \sum_1^\infty a_n F(\epsilon n) + C\epsilon KDe^{\Delta}\log\,(1-e^{-\Delta\epsilon}).$$

Hence if

$$\lim_{\epsilon \to 0} \sum_1^\infty a_n F(\epsilon n) = A,$$

we have

$$\lim_{\epsilon \to 0} \int_0^\infty F(\epsilon x)\psi(x)dx = A,$$

and under the hypotheses of our last principal theorem,

$$\int_0^\infty \psi(x)dx = \sum_1^\infty a_n = A.$$

As particular examples of this theorem, let us suppose that

$$a_n > -K/n,$$

and that either

$$\lim_{\epsilon \to 0} \sum_1^\infty a_n e^{-\epsilon n} = A$$

or

$$\lim_{\epsilon \to 0} \sum_1^\infty a_n \frac{\epsilon n e^{-\epsilon n}}{1-e^{-\epsilon n}} = A.$$

We then have

$$\sum_1^\infty a_n = A.$$

In the Hardy-Littlewood terminology, we have proved the O_L Tauberian theorems for Abel and for Lambert summation. As we have previously said, the latter theorem is equivalent to the classical theorem on the asymptotic distribution of primes.

A third theorem on summability may be formulated as follows: *Let $G(u)$ be a differentiable function assuming the value 1 at $-\infty$ and tending to 0 at $+\infty$. Let $G'(u)$ be nowhere positive, less than Ce^u, and $O(e^{-(1+\theta)u})$ at $+\infty$. Let $G(u)$ be subject to the relation*

$$G(u+l) < \theta_1 G(u) \qquad [x < T]$$

where $l<0$; $0 \leq \theta_1 < 1$. Let

$$\int_{-\infty}^\infty G(u)e^{zu}du$$

be analytic in the neighborhood of the line $R(z)=0$, with the exception of a pole of order 1 at $z=0$, and let it have no zeros on this line.

Let $\chi(x)$ be any measurable function defined for all positive arguments and satisfying the relation

$$\chi(x) > -K/\sqrt{x}.$$

Then if

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\chi(x)G(\sqrt{x}-\sqrt{N})dx = A,$$

it follows that

$$\int_{0}^{\infty}\chi(x)dx = A.$$

The truth of this theorem follows from the fact that it may be obtained from the second theorem by the following equivalences:

$$G(\log x) = F(x);$$

$$2\chi[(\log x)^2]\,\frac{\log x}{x} = \psi(x).$$

A possible value of $G(x)$ is

$$\sqrt{\frac{2}{\pi}}\int_{8}^{\infty} e^{-2t^2}d\xi.$$

Here $\displaystyle\int_{-\infty}^{\infty} G(u)e^{zu}du = \frac{1}{z}e^{z^2/16}.$

This theorem is of importance in that it indicates a mode of bringing Borel summation under the methods of this paper. Vijayaraghavan[7] has given a proof that the condition

$$e^{-x}\,\Sigma\frac{s_n x^n}{n!} \to s \qquad \left(s_n = \sum_{r=0}^{n} a_r\right)$$

together with

$$s_n > -K/\sqrt{n} \qquad (O_L)$$

leads to the conclusion that

$$\sum_{0}^{\infty} a_n = s.$$

[7] T. Vijayaraghavan, A Theorem Concerning the Summability of Series by Borel's Method. *Proc. Lond. Math. Soc.*, Series 2, 27 (1928), pp. 316–326. Cf. also a paper by the same author in *Jour. Lond. Math. Soc.*, 1 (1926), pp. 113–120.

Indeed, he shows that (O_L) may be replaced by the weaker condition (s)

$$\lim (s_m - s_n) \geq 0 \quad [m > n;\ m, n \to \infty\ ;\ n^{-\frac{1}{2}}(m-n) \to 0].$$

due to R. Schmidt. On this assumption he shows by elementary means that $s_n = O(1)$ and that

$$\lim_{x \to \infty} \frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} e^{-t^2/2x} f(t+x)\,dt = s,$$

where

$$f(y) = \begin{cases} s_n + (y-n)(s_{n+1} - s_n) & (n \leq y \leq n+1), \\ 0 & (y < 0). \end{cases}$$

His argument can be used without substantial change to show that

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-2(\sqrt{t} - \sqrt{x})^2} f(t)\,dt = s. \tag{A}$$

Instead of taking this step, Vijayaraghavan proceeds directly and in an elegant and ingenious manner to the Tauberian theorem for Borel summation itself. This is the only step in his paper which is in any intimate way concerned with the specific properties of the Borel coefficients. This step may be filled in — at least under the more stringent hypothesis (O_L) — by our third theorem. Indeed, it is not a matter of much difficulty to recast the argument in such a manner that this may be replaced by the more lenient hypothesis of Schmidt.

To show that Vijayaraghavan's assumptions imply (A) let us consider

$$\left| \frac{1}{\sqrt{2\pi x}} e^{-\frac{(t-x)^2}{2x}} - \frac{1}{\sqrt{2\pi t}} e^{-2(\sqrt{t} - \sqrt{x})^2} \right|,$$

putting

$$\sqrt{t} - \sqrt{x} = \lambda.$$

We have

$$\left| \frac{1}{\sqrt{2\pi x}} e^{-\frac{(t-x)^2}{2x}} - \frac{1}{\sqrt{2\pi t}} e^{-2(\sqrt{t} - \sqrt{x})^2} \right|$$

$$= \frac{1}{\sqrt{2\pi t}} \left| \left(1+\frac{\lambda}{\sqrt{x}}\right) e^{-\left(2\lambda^2+\frac{2\lambda^3}{\sqrt{x}}+\frac{\lambda^4}{2x}\right)} - e^{-2\lambda^2} \right|$$

$$\leq \frac{1}{\sqrt{2\pi t}} \left| \frac{\lambda}{\sqrt{x}} e^{-\lambda^2/2} + \left[\frac{2\lambda^2}{x}+\frac{\lambda^3}{x\sqrt{x}}\right] e^{-\lambda^2/2} \right| .$$

From this it follows that

$$\lim_{x\to\infty} \int_0^\infty |f(t)| \left| \frac{1}{\sqrt{2\pi x}} e^{-\frac{(t-x)^2}{2x}} - \frac{1}{\sqrt{2\pi t}} e^{-2(\sqrt{t}-\sqrt{x})^2} \right| dt$$

$$\leq \max|f(t)| \lim_{x\to\infty} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left| \frac{\lambda}{\sqrt{x}} e^{-\lambda^2/2} + \left[\frac{2\lambda^2}{x}+\frac{\lambda^3}{x\sqrt{x}}\right] e^{-\lambda^2/2} \right| d\lambda$$

$$= 0.$$

This proves formula (A).
Now, formula (A) may be written

$$s = \lim_{x\to\infty} \int_{-\infty}^{\infty} \frac{e^{-2(\sqrt{t}-\sqrt{x})^2}}{\sqrt{2\pi t}} \left[\int_0^t a_{[\xi]} d\xi\right] dt$$

$$= \lim_{x\to\infty} \int_{-\infty}^{\infty} a_{[\xi]} \int_\xi^\infty \frac{e^{-2(\sqrt{t}-\sqrt{x})^2}}{\sqrt{2\pi t}} dt$$

$$= \lim_{x\to\infty} \int_{-\infty}^{\infty} a_{[\xi]} \sqrt{\frac{2}{\pi}} \int_{\sqrt{\xi}-\sqrt{x}}^\infty e^{-2u^2} du.$$

This comes directly under our third theorem, for the $G(x)$ already discussed.

We may indeed say that practically every Tauberian theorem yet proved may be treated by the methods of this paper. In this connection, it is a matter of interest to compare the methods of Schmidt and of myself. In all cases, the types of summability handled either already are such as are treated in my second and third theorems, or may be reduced thereto by a further theorem, itself Tauberian, but of a looser character. This is the meaning of Schmidt's "gestrahlte Mittelbildungen," and of the formulae of Hardy and Littlewood invoked on pages 323 and 324 of Vijayaraghavan's paper on Borel summation.

The main Tauberian theorem — the one which puts the closest

restriction on the coefficients — is then of the type of the theorems of the present paper, save perhaps that conditions of the (O_L) type are replaced by those of the (s) type. Explicitly or implicitly, all three methods reduce this theorem to the solution of a singular integral equation

$$\int_{-\infty}^{\infty} f(x)\, K(y-x)dx = \phi(y)$$

of the first kind. Vijayaraghavan's attack depends upon the consideration of a finite or infinite set of equations which may be obtained from this by successive differentiation. Schmidt attacks this problem more systematically by the method of moments, which yields a general answer, but an answer not always easy of concrete application, as in the case of Lambert summation. The present paper uses the method of Fourier analysis, also of general bearing, and in the opinion of the author, simpler. The divergence of standpoints here reflected is but a reflection of that prevailing in the operational calculus at large. It is in that more general region that the relative merits of the different methods must be decided. The present author[8] has already given grounds for believing that the power of Fourier methods is here of the same order as that of methods depending on successive differentiation or on the theory of moments and of the Laplace transformation and has succeeded in using them to establish the asymptotic formulae of Heaviside. He is therefore inclined to look favorably upon them in this closely related field.

At all events, Tauberian theorems have undergone a remarkable change of status since the early days of pioneer work, in which four or five years of brilliant technique and persistent effort were often necessary to reduce an O condition to an o form. All three of the present methods are in essence reasonably simple, and at least two of them seem to be of sufficient generality for all our present needs. What the reason for this change may be is perhaps hard to say; the fairest explanation is perhaps that Hardy and Littlewood[9] had already succeeded in casting a large group of Tauberian theorems into the form of the equivalence of two means of positive quantities.

[8] N. Wiener, The Operational Calculus. *Math. Ann.* 95 (1926), pp. 557–584.
[9] Loc. cit.

NOTE ON CONGRUENCES IN RIEMANNIAN SPACE

By C. L. E. Moore

1. It is the purpose of this note to extend to Riemannian 3-space the well known property of congruences of straight lines in ordinary Euclidean 3-space. "Through each line of a congruence passes two developable surfaces, whose generators are lines of the congruence. The necessary and sufficient condition that the congruence be normal is that these developable surfaces intersect orthogonally." For Euclidean space these conditions apply throughout the whole length of a line of the complex, but in Riemannian space, since we have no analog of a plane, we must limit ourselves to a planar element so that the corresponding conditions hold at a point and not along the whole curve.

In Euclidean space if two straight lines intersect, a vector which cuts both of them when displaced along one of them, parallel to itself, will continue to cut both. This is the property of developable surfaces which is used. In other words we replace the notion of tangent plane to a developable by that of planar elements having the above property of planar elements of a developable surface. We call these *developable planar elements*. For geodesic congruences we then have the following theorem: *Through each line element of a real geodesic congruence passes two developable planar elements and the necessary and sufficienct condition that the congruence be normal is that these planar elements everywhere intersect at right angles.*

For general congruences we find that the directions which when moved by parallelism along the element of the congruence do generate a developable planar element form a quadric cone which has the congruence direction as an element. We prove that *the necessary and sufficient condition that the congruence be normal is that this cone contain two generators perpendicular to the element of the congruence and to each other.*

2. *Geodesic Congruences in 3-Space.* Let the geodesic congruence be one set of coördinate lines, say, x^3. The element of arc is then

$$ds^2 = a_{rs}dx^r dx^s \tag{1}$$

where[1] $$a_{33}=1, \quad \frac{\partial a_{13}}{\partial x_3}=\frac{\partial a_{23}}{\partial x_3}=0.$$

The coördinate surfaces $x^3=$ const. can be chosen as any one parametered set of real surfaces. That is, we can choose these surfaces so that

$$a_{11}a_{22}-a_{12}^2 \neq 0. \tag{2}$$

With the element of arc as chosen, we see at once that

$$\left\{ \begin{matrix} 3 & 3 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} 3 & 3 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 3 & 1 \\ 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 & 2 \\ 3 \end{matrix} \right\} = 0, \tag{3}$$

and therefore the increment added to a vector by parallel displacement along one of the geodesics is independent of the x^3 component of that vector. We will then take for our vector to be displaced, one of directions $(dx^1, dx^2, 0)$ tangent to the x^3-coördinate surface passing through the point considered. Then by the formulae for parallel displacement we have

$$\delta x^1 = \left\{ \begin{matrix} 1 & 3 \\ 1 \end{matrix} \right\} dx^1\, dx^3 + \left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} dx^2\, dx^3$$

$$\delta x^2 = \left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} dx^1\, dx^3 + \left\{ \begin{matrix} 2 & 3 \\ 2 \end{matrix} \right\} dx^2\, dx^3$$

$$\delta x^3 = 0.$$

If the new vector is to cut the same two near by geodesics as the original vector $(dx^1, dx^2, 0)$, then

$$\frac{\delta x_2}{\delta x_1} = \frac{\left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} dx^1 + \left\{ \begin{matrix} 2 & 3 \\ 2 \end{matrix} \right\} dx^2}{\left\{ \begin{matrix} 1 & 3 \\ 1 \end{matrix} \right\} dx^1 + \left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} dx^2} = \frac{dx^2}{dx^1}.$$

If we put $\lambda = \dfrac{dx^2}{dx^1}$, this becomes

1 Eisenhart, Riemannian Geometry, page 56.

$$\left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} \lambda^2 + \left(\left\{ \begin{matrix} 1 & 3 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 2 & 3 \\ 2 \end{matrix} \right\} \right) \lambda - \left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} = 0. \tag{4}$$

The developable planar element is determined by the vectors $(dx^1, dx^2, 0)$ and $(0, 0, dx^3)$. Hence *through each line element of the geodesic congruence passes two developable planar elements.*

If we denote the two directions determined from (4) by $(1, \lambda_1, 0)$ and $(1, \lambda_2, 0)$ where λ_1 and λ_2 are the roots of (4), the covarient coördinates of the normals to the two developable planar elements are

$$(\lambda_1, -1, 0) \text{ and } (\lambda_2 - 1, 0).$$

The condition that the two developable elements be orthogonal is that the last named vectors be perpendicular. That is

$$a^{11}\lambda_1\lambda_2 - a^{12}(\lambda_1 + \lambda_2) + a^{22} = 0.$$

From (4) this becomes

$$-a^{11}\left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} + a^{12}\left(\left\{ \begin{matrix} 1 & 3 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 2 & 3 \\ 2 \end{matrix} \right\} \right) + a^{22}\left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} = 0.$$

If we substitute the values of the Christoffel symbols in terms of the coefficients of (1), this reduces to

$$[(a^{12})^2 - a^{11}a^{22}]\left(\frac{\partial a_{23}}{\partial x^1} - \frac{\partial a_{13}}{\partial x^2} \right) = 0,$$

and since the first factor does not vanish we have for the condition of perpendicularity,

$$\frac{\partial a_{23}}{\partial x^1} - \frac{\partial a_{13}}{\partial x^2} = 0. \tag{5}$$

The covariant coördinates of the contravariant vector $(0, 0, 1)$ tangent to lines of parameter x^3 are $(a_{13}, a_{23}, 1)$ and the condition that this last be the coördinates of a normal congruence is (5). Hence, *the necessary and sufficient condition that a congruence of geodesics be normal is that the two developable planar elements passing through each line element of the congruence be perpendicular.*

3. In Euclidean space the developable surface elements are always surface building, but in Riemannian space this is not the case. Let us now investigate the condition that in a normal con-

gruence these elements be surface building. Since the geodesic congruence is normal, (1) can be written in the form (6)

$$ds^2 = a_{11}(dx^1)^2 + 2a_{12}dx^1\,dx^2 + a_{22}(dx^2)^2 + (dx^3)^2 \tag{6}$$

and the covariant coördinates of the vector perpendicular to one of the developable elements are $(\lambda_1, -1, 0)$ and the condition that the surface element be surface building is the same as the condition that these vectors form a normal congruence. That is

$$\frac{\partial \lambda_1}{\partial x^3} = 0. \tag{7}$$

This means that as the element is moved by parallelism along a line of the geodesic congruence it will always cut the same two geodesics. If then on one surface we choose the lines determined by (4) as parameter lines

$$a_{12} = 0, \quad \left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} = 0,$$

and from (7) this must hold for every point of space and the element of arc takes the simple form

$$ds^2 = a_{11}(dx^1)^2 + a_{22}(dx^2)^2 + (dx^3)^2. \tag{8}$$

The developable elements will envelope the coördinate surface $x^1 = \text{const.}$, $x^2 = \text{const.}$ A tangent plane to one of these surfaces is moved by parallelism along the geodesic generator into a tangent plane and the surfaces are therefore of the parabolic type. The three sets of coördinate surfaces form a triply orthogonal system. Then *the necessary and sufficient condition that a geodesic congruence can form the intersection of two sets of surfaces of a triply orthogonal system is that the congruence be normal and that the developable elements be surface building.*

4. If the congruence is normal and on a surface $x^3 = k$, we choose the directions defined by (4) for coördinates on this surface we have

$$a_{12} = 0. \quad \left\{ \begin{matrix} 1 & 3 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 & 3 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 2 \\ 3 \end{matrix} \right\} = 0.$$

The second fundamental form[2] for this surface is

$$\frac{1}{\sqrt{a_{12}^2-a_{11}a_{22}}}\left[\begin{Bmatrix}1&1\\3\end{Bmatrix}(dx^1)^2+2\begin{Bmatrix}1&2\\3\end{Bmatrix}dx^1dx^2+\begin{Bmatrix}2&2\\3\end{Bmatrix}(dx^2)^2\right].$$

But from the relation above we see that the product term vanishes and hence the coördinate lines are lines of curvature. Hence, *the surface element formed by a line of curvature and a normal is a developable element.* It is well known that directions normal to a surface at two near by points of a line of curvature lie in a plane and now we have the added property that if a tangent to a line of curvature is moved by parallelism along a normal it will cut two near by normals in its new position.

5. *General Congruences.* We can also apply the foregoing notions to any congruence in a Riemannian 3-space. If any congruence of curves is defined in contravariant coördinates by λ^i, the condition that an element ds, cutting a curve of the congruence λ^i and the curve infinitesimally near, will do the same when displaced by parallelism along λ^i, can be expressed by saying that λ^i, $\lambda^i+\delta\lambda^i$, ds all lie in the same plane. This may be written

$$\epsilon_{ijk}\lambda^i\,\delta\lambda^j dx^k=0, \tag{9}$$

where $\delta\lambda^i$ indicates the covariant differential and dx^k are the components of ds. Since

$$\delta\lambda^i=d\lambda^i+\begin{Bmatrix}j&k\\i\end{Bmatrix}\lambda^j dx^k$$

we see that (9) is quadratic in dx^k. Then associated with the element λ^i through each point P there are an infinite number of developable surface elements, and the directions which moved along λ^i to generate these developable elements, generate a quadratic cone, on which λ^i is a generator.

The character of the congruence is largely indicated by this cone. We can simplify the calculations by choosing λ^i as one set of coördinate lines, x^3 say, and also let x^3 be the arc length along these curves. Then

$$ds^2=a_{rs}dx^r dx^s,\quad a_{33}=1. \tag{10}$$

[2] Eisenhart, loc. cit., page 150.

Equation (9) then becomes

$$\left\{\begin{matrix}1&3\\2\end{matrix}\right\}(dx^1)^2+\left[\left\{\begin{matrix}2&3\\2\end{matrix}\right\}-\left\{\begin{matrix}1&3\\1\end{matrix}\right\}\right]dx^1dx^2-\left\{\begin{matrix}2&3\\1\end{matrix}\right\}(dx^2)^2$$
$$+\left\{\begin{matrix}3&3\\2\end{matrix}\right\}dx^1dx^3-\left\{\begin{matrix}3&3\\1\end{matrix}\right\}dx^2\,dx^3-0. \tag{11}$$

The directions on this cone which are perpendicular to λ^i, *i.e.*, to the vector (0, 0, 1), must also satisfy

$$a_{13}dx^1+a_{23}dx^2+dx^3=0. \tag{12}$$

Eliminating dx^3 between (11) and (12)

$$\left[\left\{\begin{matrix}1&3\\2\end{matrix}\right\}-a_{13}\left\{\begin{matrix}3&3\\2\end{matrix}\right\}\right](dx^1)^2+\left[\left\{\begin{matrix}2&3\\2\end{matrix}\right\}-\left\{\begin{matrix}1&3\\1\end{matrix}\right\}-a_{23}\left\{\begin{matrix}3&3\\2\end{matrix}\right\}\right.$$
$$\left.+a_{13}\left\{\begin{matrix}3&3\\1\end{matrix}\right\}\right]dx^1dx^2+\left[a_{23}\left\{\begin{matrix}3&3\\1\end{matrix}\right\}-\left\{\begin{matrix}2&3\\1\end{matrix}\right\}\right](dx^2)^2=0. \tag{13}$$

Let the directions determined by the solution of (13) be

$$(\lambda,\ 1,\ -a_{13}\lambda-a_{23}),\quad (\mu,\ 1,\ -a_{13}\mu-a_{23}).$$

These two vectors are perpendicular if

$$(a_{11}-a_{13}^2)\lambda\mu+(a_{12}-a_{13}a_{23})(\lambda+\mu)+a_{22}-a_{23}^2=0. \tag{14}$$

Substituting for $\lambda\mu$, $\lambda+\mu$ from (13) this reduces to

$$a^{13}\left\{\begin{matrix}3&3\\2\end{matrix}\right\}-a^{23}\left\{\begin{matrix}3&3\\1\end{matrix}\right\}+a^{11}\left\{\begin{matrix}1&3\\2\end{matrix}\right\}-a^{22}\left\{\begin{matrix}2&3\\1\end{matrix}\right\}$$
$$-a^{12}\left(\left\{\begin{matrix}1&3\\1\end{matrix}\right\}-\left\{\begin{matrix}2&3\\2\end{matrix}\right\}\right)=0 \tag{15}$$

where a^{ij} has the usual meaning. If now we substitute for the Christoffel symbols, (15) can finally be reduced to

$$a_{33}\frac{\partial a_{13}}{\partial x_3}-a_{13}\frac{\partial a_{23}}{\partial x_3}-\frac{\partial a_{13}}{\partial x_2}+\frac{\partial a_{23}}{\partial x_1}=0, \tag{16}$$

which is at once seen to be the condition that the congruence be normal. Hence *the necessary and sufficient condition that a congruence be normal is that the cone* (9) *cut the plane element per-*

pendicular to λ^i in perpendicular directions. Or, *the developable elements determined by directions normal to λ^i must be perpendicular.*

If (16) is satisfied and we take the orthogonal surfaces as one set of coördinate surfaces (a_{13} cannot now be unity) from paragraph 4 it is at once seen that the directions in which (9) cuts the tangent planes of these surfaces are along the lines of curvature. The congruences formed by the lines of curvature of the normal surfaces are the congruences conical with the given congruences.[3]

6. Since the cone (9) contains λ^i when it degenerates it will degenerate into two plane pencils, one of which must contain λ^i, this will mean that to this pencil of directions correspond the same developable surface element. In case of a Euclidean 3-space this condition would mean that λ^i is a congruence of plane curves. If both pencils contain λ^i then the congruence will be a congruence of geodesics.

7. *Extensions to n-space.* In order to generalize the foregoing to n-dimensions we first consider a geodesic congruence. Let x^n be this congruence, then

$$\left\{ \begin{matrix} i & n \\ & n \end{matrix} \right\} = 0 \tag{17}$$

and equation (9) takes the form

$$\frac{\left\{ \begin{matrix} i & n \\ & k \end{matrix} \right\} dx^i}{\left\{ \begin{matrix} i & n \\ n-1 & \end{matrix} \right\} dx^i} = \frac{dx^k}{dx^{n-1}} . \tag{18}$$

It is to be noted that dx^n does not occur and so this is a well-known set of equations and the number of solutions is $n-1$. Consequently through each linear element passes $n-1$ developable planar elements. If (17) does not hold, that is, if the congruence is a general congruence, then (18) will have dx^n on the left side and hence will represent a cone. If we substitute for dx^n any linear function of $dx^1, dx^2, \cdots dx^{n-1}$ this set of equations will again have $n-1$ solutions, which means that any plane through the

[3] Ricci Levi-Civati, Méthodes de calcul differentiel absolu, *Math. Ann.*, Vol. 54, p. 156.

vertex will cut $n-1$ generators from the cone, hence the cone is of degree $n-1$.

We will then state the following theorem without giving the details of the proof. *The developable elements passing through any linear element of a congruence in n-dimensions are determined by the tangent to the congruence passing through the given point and an element of a cone of order* $n-1$. *The necessary and sufficient condition that the congruence be normal is that the elements of the cone cut by a hyperplane perpendicular to the direction of the congruence, are mutually orthogonal. If the cone factors into* $n-1$ *linear factors, each containing the congruence direction, then the congruence must be a geodesic congruence. If only one of the plane factors contains the congruence direction, then the congruence is the analog of a congruence of curves which lie in an* $(n-1)$*-linear space. If two of the linear factors contain the congruence direction, each curve of the congruence is the analog of a curve which lies in an* $(n-2)$*-space, etc.*

ON SETS OF PRINCIPAL DIRECTIONS IN A RIEMANNIAN MANIFOLD OF FOUR DIMENSIONS

By D. J. STRUIK

§1. Riemann-Christoffel Tensor and Quadratic Complex. The Riemann-Christoffel curvature tensor $K_{\omega\mu\lambda\nu}$ of a four-dimensional Riemannian manifold V_4 corresponds in a projective space of three dimensions P_3 to a quadratic line complex

$$K_{\omega\mu\lambda\nu}p^{\omega\mu}p^{\lambda\nu}=0. \tag{1}$$

Here the $p^{\omega\mu}$ are simple bivectors, representing in P_3 the Plücker coördinates of the lines of the complex. They satisfy the relation

$$p^{12}p^{34}+p^{13}p^{42}+p^{14}p^{23}=0 \tag{2}$$

or, in a shorter notation:

$$p^{[\omega\mu}p^{\lambda\nu]}=0.^{1} \tag{3}$$

The resulting relation

$$K_{[\omega\mu\lambda\nu]}=0 \tag{4}$$

is fulfilled as a result of the identities existing for the curvature tensor.[2]

Klein has shown in his dissertation[3] how the equation of the quadratic complex in P_3 can be cast into a special form by an

[1] The brackets mean the alternation over the enclosed indices, *e.g.*,

$$v_{[\lambda\mu]}=\tfrac{1}{2}(v_{\lambda\mu}-v_{\mu\lambda})$$

$$p_{[\lambda\mu}q_{\nu]}=\tfrac{1}{3}(p_{\lambda\mu}q_{\nu}+p_{\mu\nu}q_{\lambda}+p_{\nu\lambda}q_{\mu}-p_{\mu\lambda}q_{\nu}-p_{\nu\mu}q_{\lambda}-p_{\lambda\nu}q_{\mu}).$$

[2] Comp. *A. Clebsch.* Ueber die Plückerschen Complexe. Mathematische Annalen 2 (1870), p. 1–8.

[3] *F. Klein.* Ueber die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Koordinaten auf eine canonische Form (Inauguraldissertation Bonn, 1868; Mathematische Annalen 23 (1884), p. 539–578; Gesammelte mathematische Abhandlungen I (1921), p. 5–49). Comp. also *C. M. Jessop.* A treatise on the line complex (Cambridge, University Press), 1903, p. 97 or *K. Zindler*, Algebraische Liniengeometrie, p. 1015, in (Pascal's) Repertorium der hoheren Mathematik II, 2, Raumgeometrie (Teubner, Leipzig-Berlin), 1922).

Comp. also *C. L. E. Moore* and *H. B. Phillips.* The dyadics which occur in a point space of three dimensions. Proc. American Academy of Arts and Sciences 53 (1917–18), p. 389–438, esp. p. 420.

imaginary transformation of the Plücker coördinates. We show in this paper how a similar transformation in V_4 detects particular sets of preferred directions of the V_4, introduced for the first time by Kretschmann[4]).

§2. **The principal bivectors of V_4.** We start with the covariant components of the curvature tensor $R^{\omega\mu\lambda\nu}$ and ask whether there are covariant bivectors $p_{\lambda\nu}$ for which

$$K_{\omega\mu[\lambda\nu}p_{\rho\sigma]} = \lambda p_{\omega\mu}E_{[\lambda\mu\rho\sigma]}. \tag{5}$$

Here λ is a coefficient and $E_{[\lambda\mu\rho\sigma]} = E_{\lambda\mu\rho\sigma}$ the unit quadruvector of the V_4 with components ± 1 when the indices are not equal ($+1$ for even, -1 for odd permutations) and 0 when at least two indices are equal. The $p_{\lambda\mu}$ need not be simple, that is, the equation of type (3) is not necessarily fulfilled.

Equation (5) leads to a set of six equations, of which the two first ones are:

$$\begin{aligned} &K_{1212}p_{34} + K_{1213}p_{42} + K_{1214}p_{23} + K_{1223}p_{14} + K_{1224}p_{31} + K_{1234}p_{12} = \lambda p_{12} \\ &K_{1312}p_{34} + K_{1313}p_{42} + K_{1314}p_{23} + K_{1323}p_{14} + K_{1324}p_{31} + K_{1334}p_{12} = \lambda p_{13}. \end{aligned} \tag{6}$$

The solution depends on the determinant equation:

$$\begin{vmatrix} K_{1212} & -K_{1213} & K_{1214} & K_{1223} & -K_{1224} & K_{1234}-\lambda \\ K_{1312} & -K_{1313} & K_{1314} & K_{1323} & -K_{1324}-\lambda & K_{1334} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{3412}-\lambda & -K_{3413} & \cdot & \cdot & \cdot & \cdot & K_{3434} \end{vmatrix} = 0 \tag{7}$$

We will only consider the case that this equation has six different roots $\lambda_1, \lambda_2, \cdots, \lambda_6$. We then have six different bivectors

$$p^1_{\omega\mu},\ p^2_{\omega\mu},\ \cdots,\ p^6_{\omega\mu}$$

fulfilling equation (5). We can normalize them by assuming

$$p^i_{[\omega\mu}\,p^i_{\lambda\nu]} = E_{\omega\mu\lambda\nu} \tag{8}$$

and call them the *principal bivectors* of $K_{\omega\mu}$. They fulfil the rela-

[4] *E. Kretschmann.* Über den physikalischen Sinn der Relativitätspostulate. Annalen der Physik 53 (1917), p. 575–614. Comp. also *R. Bach.* Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs. Mathematische Zeitschrift 9 (1921), p. 110–135.

tions, following from (5):

$$p^i_{[\omega\mu}p^j_{\lambda\nu]}=0, \quad i\neq j \tag{9}$$

and correspond, in P_3, to a set of linear complexes in involution.

Referred to these bivectors $K_{\omega\mu\lambda\nu}$ assumes the form

$$K_{\omega\mu\lambda\nu}=\sum_{j}^{1,\cdots,6}\lambda_j p^j{}_{\omega\mu}p^j_{\lambda\nu}. \tag{10}$$

§3. **Sets of principal directions.** The principal bivectors are in general not simple. Each pair of them, however, determines two simple bivectors by the equation:

$$(p^i_{[\omega\mu}+\alpha p^j_{\omega\mu])}\,(p^i_{\lambda\nu}+\alpha p^i_{\omega\mu])}=0, \qquad i\neq j \tag{11}$$

where the alternation is on ω, μ, λ, ν. On account of (8) and (9) this gives

$$\alpha=\pm i=\pm\sqrt{-1}. \tag{12}$$

We can divide the six principal bivectors into 15 pairs with $i\neq j$, which can be arranged in 15 different ways to sets of three pairs containing all 6 indices (as 12, 34, 56; 13, 46, 25). Each pair determines 2 simple bivectors. We thus find 15 different sets of simple bivectors determined by equations (11). We call one set of them $\pi^i_{\omega\mu}$, $i=1,\cdots,6$ and arrange them (dropping the $\omega\mu$ for a while):

$$\begin{array}{c|c|c} \pi^1=p^1+ip^2 & \pi^2=p^3+ip^4 & \pi^3=p^5+ip^6 \\ \pi^4=p^1-ip^2 & \pi^5=p^3-ip^4 & \pi^6=p^5-ip^6 \end{array} \tag{13}$$

Then we have the relations, following from (8) and (9)

$$\begin{aligned} &\pi^1{}_{[\omega\mu}\,\pi^4{}_{\lambda\nu]}=2E_{\omega\mu\lambda\nu} \\ &\pi^1{}_{[\omega\mu}\,\pi^a{}_{\lambda\nu]}=0, \quad a=2,3,5,6 \\ &\text{etc.} \end{aligned} \tag{14}$$

This means that we can introduce four linear independent covariant vectors v^k, $k=1, 2, 3, 4$ in such a way that the π^i are of the form

$$v^i_{[\omega}v^j_{\mu]}, \quad i\neq j, \quad i, j=1, 2, 3, 4.$$

Passing to contravariant components it means that there are four independent directions issuing from the point of V_4 in consideration, whose 6 connecting planes are the planes of the contravariant bivectors π^i. If we introduce these vectors into equation (10), we find an equation of the form

$$\begin{aligned} K_{\omega\mu\lambda\nu} = {} & K_{1212}\, v^1_{[\omega}\, v^2_{\mu]}\, v^1_{[\lambda}\, v^2_{\nu]} + K_{1313} v^1_{[\omega}\, v^2_{\mu]}\, v^1_{[\lambda}\, v^2_{\nu]} + \\ & + \cdots + K_{3434}\, v^3_{[\omega}\, v^4_{\mu]}\, v^3_{[\omega}\, v^4_{\mu]} + \\ & + K_{1234} v^1_{[\omega}\, v^2_{\mu]}\, v^3_{[\lambda}\, v^4_{\nu]} + K_{1234}\, v^1_{\omega}\, v^3_{\mu}\, v^2_{\lambda}\, v^4_{\mu} + \\ & + K_{1423} v^1_{[\omega}\, v^4_{\mu]}\, v^2_{[\lambda}\, v^3_{\mu]} \\ & + K_{1234} v^3_{[\omega}\, v^4_{\mu]}\, v^1_{[\lambda}\, v^2_{\nu]} + \cdots, \end{aligned} \tag{15}$$

with

$$K_{1212} = K_{3434}; \quad K_{1313} = K_{2424}; \quad K_{1414} = K_{2323} \tag{16}$$

The coefficients K_{ijkl} of (16) and the right member of (15) have a meaning different from the $K_{\omega\mu\lambda\nu}$ of the left side of (15) and (10); they represent the components of the curvature tensor with respect to the set of v^k.

This scheme (15)

	12	13	14	23	24	34
12	K_{1212}	0	0	0	0	K_{1234}
13	0	K_{1313}	0	0	K_{1234}	0
14	0	0	K_{1414}	K_{1423}	0	0
23	0	0	K_{2314}	K_{2323}	0	0
24	0	K_{2413}	0	0	K_{2424}	0
34	K_{1234}	0	0	0	0	K_{3434}

has been found for the first time by *Kretschmann* by a simple enumeration of constants. In the case that equation (7) has different roots, such a scheme can be obtained in 15 different ways, yielding different K_{ijkl}.

The directions determined by the vectors v^i_ω can be called a

set of *principal directions* of the Riemann Christoffel tensor. They are, in general, not orthogonal.

§4. **Relation to the principal directions of Ricci.** Let $g^{\lambda\mu}$ be the covariant fundamental tensor of the V_4. We then find from (15) for the contracted curvature tensor:

$$\begin{aligned} K_{\omega\nu} = K_{\omega\mu\lambda\nu} g^{\mu\lambda} = &-\tfrac{1}{4} v_\omega{}^1 v_\nu{}^1 \left(K_{1212} v_\mu{}^2 v_\lambda{}^2 g^{\mu\lambda} + \right. \\ &\left. + K_{1313} v^3_\mu v_\lambda{}^3 g^{\mu\lambda} + K_{1414} v_\mu{}^4 v_\lambda{}^4 g^{\mu\lambda}\right) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ \tfrac{1}{4} v_\omega{}^1 v_\nu{}^2 \left(K_{1212} v_\mu{}^1 v_\lambda{}^2 g^{\mu\lambda} - \right. \\ &\left. - 2 K_{1234} v_\mu{}^3 v_\lambda{}^4 g^{\mu\lambda} - 2 K_{1423} v_\mu{}^3 v_\lambda{}^4 g^{\mu\lambda}\right) \\ &+ \cdots \end{aligned} \tag{17}$$

This shows that when the principal directions v^i_ω, $i = 1, 2, 3, 4$, are mutually orthogonal, the terms in v^i_ω, v^j_ν $i \neq j$, vanish:

If a set of principal directions of the Riemann-Christoffel tensor is orthogonal, it coincides with the set of principal directions of the contracted curvature tensor (the so-called principal directions of Ricci).

DEFORMATION OF PLANE PIPES

By Professor William Hovgaard

A. Introduction

The present paper is a continuation of that published in Vol. VI, No. 2, of the Journal of Mathematics and Physics, November, 1926.[1] It is the first installment of a record, which it is intended to make of further study and experimentation in the field of pipe bends. The paper shows how to determine deflections and reactions in steam pipe lines of various lay-out, comprising different types of bends. It is essentially of a fundamental nature, giving a development on first principles of general formulas applicable to all cases. On account of the great diversity in pipe bends and in the lay-out of piping, it is believed to be of primary value to the engineer to possess the master key to all problems, rather than the solution of a number of special problems, many of which he may never meet. It is therefore considered important first to secure a full understanding of the methods and of the assumptions involved, after which ready-made handy formulas, tables and diagrams can be used more easily and safely.

The paper does, however, give the solution for several special types of bends commonly used in practice, but chiefly for the purpose of illustrating the general method and principles. In one case it is attempted to develop approximate formulas. Algebraic and graphical methods of solution are described, in particular for evaluating the integrals that occur in the formulas.

In a later paper it is intended to give a report and analysis of tests on pipe bends carried out at the Massachusetts Institute of Technology in the early part of 1928, possibly supplemented by the results of further tests. The tests referred to were carried out on a 9½ and a 14½-inch pipe bend, larger than any hitherto tested and gave extremely interesting results. The longitudinal strain measurements were carried out directly by strain gages as

[1] Subsequently reprinted in the Bulletin of the Massachusetts Institute of Technology, Vol. 62, No. 64, and here referred to as the "first paper" or the "1926 paper."

in the 1926 tests, but the transverse strains were calculated from the change of curvature obtained from measurement of a number of diameters around a section, and this new method proved entirely successful. Another new feature in these tests was that they were carried to destruction, making it possible to determine with fair accuracy the limit of proportionality in the deformation of the pipe as a whole, and it was attempted to correlate this limit with the calculated and measured stresses. Finally a close study was made of the Lüders lines, which began to appear practically at the limit of proportionality. As the analysis of these tests is not quite completed and as it is hoped to verify the results by further tests in a near future, it was thought best to defer a detailed report and discussion of them to a later paper, the more so as the paper would otherwise become rather voluminous.

As many readers of the present paper may not be acquainted with the 1926 paper, a brief review of it is here given in order to provide the proper background and perspective for the following discussion.

Review of the First Paper

When a curved pipe is subject to bending, the circular section tends to take a flattened oval form, whereby the stresses in the material farthest from the neutral axis are relieved. The stresses, instead of following the usual linear law, will be so distributed as to have their maximum values nearer to the neutral axis; and the angular deflection of the bend will be increased before equilibrium with a given bending moment is established. This was pointed out already in 1911 by Dr. v. Karman,[2] who developed formulas for the augmented deflections. These formulas remained unconfirmed for a number of years and were by some contested, by others ignored. In his first paper the author proved the correctness of Dr. v. Karman's formulas by an entirely different method and published the results and the analysis of a series of tests made at the Massachusetts Institute of Technology as well as an analysis of tests previously made by Professor Bantlin, which fairly conclusively showed the correctness of v. Karman's theory.[3]

[2] Zeitschr. Vereines Deutscher Ingenieure, 1911, Vol. 55, p. 1889.
[3] This theory has been further corroborated by the 1928 tests.

The following notation, which was used in the first paper, will also be used here:

M = bending moment.
R = radius of curvature of axis of pipe.
$2h$ = wall thickness of pipe.
r = radius of the middle surface of the pipe wall.
I = moment of inertia of the area of the transverse section of the pipe wall about the neutral axis.
E = modulus of elasticity.
$d\psi$ = the angle of a small arc of the pipe axis comprised between two consecutive transverse sections.
$\Delta d\psi$ = the change in $d\psi$ consequent on the flexural strain.
s = the length or girth of the axis reckoned from some point of reference to the section under consideration.

According to the ordinary approximate theory we have:

$$\frac{\Delta d\psi}{d\psi} = \frac{MR}{EI} \quad \text{or} \quad \Delta d\psi = \frac{M}{EI} ds \tag{1}$$

which are the formulas commonly used for curved bars, but in the first paper it was shown that due to flattening of the section the angular deflection of an elemental piece of a curved pipe was actually:

$$\frac{\Delta d\psi}{d\psi} = K \frac{MR}{EI} \quad \text{or} \quad \Delta d\psi = \frac{KM}{EI} ds \tag{2}$$

where

$$K = \frac{48h^2R^2 + 10r^4}{48h^2R^2 + r^4}. \tag{3}$$

The value of this coefficient depends on the relative values of h, r and R. Karman put $\frac{2hR}{r^2} = \lambda$ which makes:

$$K = \frac{12\lambda^2 + 10}{12\lambda^2 + 1}. \tag{4}$$

He obtained his formulas by making the potential elastic energy a minimum and included in the expression for the energy not only the longitudinal but also the transverse strains.

The coefficient K has the same effect as if the bending moment were increased, since K is greater than one. In fact, K varies from unity in straight pipes to 4 or 5 in pipes bent to a sharp curve. K occurs again as a simple factor in the expressions for longitudinal and transverse stresses p_1 and p_2 and in the displacements of the pipe due to its deformations. It might also be regarded as associated with the moment of inertia, reducing it to $\frac{I}{K}$.

In the tests on pipe bends carried out at the Massachusetts Institute of Technology in 1926, strain measurements of the pipe wall were carried out for the first time and as explained in the paper showed a close agreement with the formulas.

It was shown also, both theoretically and experimentally, that the transverse stresses occurring at the end of the major and minor axis of the deformed transverse section, might under certain circumstances be much greater than the longitudinal stresses, a fact which had not hitherto been taken into account.

Formulas were developed for determining the deflections in lyre-shaped bends and a graphical method was presented by which all forms of pipe bends could be dealt with. It was shown that for the purpose of analyzing experimental results the algebraic method was often unsatisfactory, because bends can seldom be made of exact geometrical forms.

B. Calculation of Forces, Moments and Deflections in Pipe-Bends.

1. *General.* The general lay-out of steam piping is governed primarily by practical considerations, which in most cases determine the total length as well as the changes in direction and level of the piping as dictated by local conditions. The designer has to conform to these requirements and at the same time arrange the piping so as to secure the flexibility necessary to provide for elongations and contractions due to changes in temperature. He must also arrange the piping so as to permit efficient drainage of the water of condensation; he must provide intervals of sufficient strength, and supports which will permit easy movements of the pipe by changes of temperature.

The discussion will be limited to a consideration of the flexi-

bility and deflections of the piping, as well as the forces and moments called forth by a rise in temperature. The practical requirements to anchorage, support, and drainage are considered to be outside the scope of this paper. Only pipes in one plane will be considered. The question of strength and stresses will be discussed in a later paper.

Before dealing with concrete cases we shall first see how the fundamental formulas are obtained which connect the deflections and displacements of a pipe-bend with the terminal forces and couples. The problem is indeed not a new one, in fact, the general method of solution is given in textbooks on Applied Mechanics, but for engineers who have not made a special study of this subject and who desire to refresh their memory, it is convenient to have the development of the required formulas placed before them, presented in the same symbols as used hereafter, and prepared especially with a view to a solution of the problems occurring in pipe bends.

The term "deflection" is here used to cover any change of position, linear or angular, due to deformation, but the terms "displacement" and "rotation" are used where it is desired to be more specific, especially to distinguish between linear and angular deflections, and where deflections are caused by heat expansion.

2. **Fundamental Formulas for a Plane Pipe.** Consider the problem in its most general form. A pipe bent to any shape in one plane, as shown in Fig. 1, is anchored at the ends O and A. The anchorages are such as to prevent linear displacements of the ends, but may or may not prevent rotation. In the former case we say that the pipe is "fixed" at the ends, in the latter case that it is "hinged." In many cases one end is "fixed," while the other may be regarded as "hinged." Finally one end may be "fixed" while the other is partly free, being capable of rotation and displacement in one direction, in which case we speak of the end of the pipe as being "restrained" or "guided."

For the purpose of mathematical analysis we first suppose that the pipe is fixed at O, but perfectly free to move at A, where there is no restraint and no external forces acting. Now a rise in temperature takes place and the pipe expands. The A-end will be

displaced by a certain amount, the components of which in direction of the coördinate axes, Δx_A and Δy_A, have values, which can be readily calculated as explained below.

If the A-end is hinged instead of being free, the displacements Δx_A and Δy_A will be zero, but the pipe will suffer deformation and generally the tangent at A will rotate through a small angle $\Delta\psi_A$. Reaction forces will be called into play at both ends and a reaction couple will be created at O, but no couple will exist at the hinged end A.

The result is the same as if the pipe had been first allowed to expand and then forced back by extraneous forces equal to the reaction forces, so that the displacements would first be $+\Delta x_A$ and $+\Delta y_A$ and then $-\Delta x_A$ and $-\Delta y_A$.

If both ends are fixed, we may again conceive the process to consist of two steps, during the first of which the pipe expands by heating, but in order to force the A-end back to its original position we must now apply a couple besides the forces, and the A-end has to be rotated forcibly through an angle $-\Delta\psi_A$.

Let the reaction forces acting at A be P' and Q' and the reaction couple M_A, while at O the forces are P and Q and the couple M_0. Since these forces and couples form a system in equilibrium we must have

$$\left.\begin{aligned} P' &= -P, \qquad Q' = -Q \\ M_A &= -(M_0 + PH - QL \end{aligned}\right\} \tag{7}$$

where L and H are the coördinates of A.

It is to be noted that all the terminal forces and couples with which we deal here are *reactions* exerted by the supports on the pipe.

Generally we are left with three unknown quantities P, Q and M_0, but when the pipe is hinged at both ends and in cases of symmetry, one or two of these may be equal to zero.

For the determination of these unknown quantities we employ the method of deflection by which the component displacements and the rotation at A are expressed in terms of the three unknown reactions. We have then three equations and six unknown quantities, but in accordance with the foregoing we shall assume that the displacements Δx_A and Δy_A produced by the reactions are

equal to those produced by the heat expansion although in opposite directions and since the latter are known we may substitute them for Δx_A and Δy_A with negative sign. The expression for the rotation enables us to eliminate one more of the unknown quantities.

Strictly the expansion at A due to heating is not exactly equal to the displacement obtained by applying the reaction forces, because we assume these forces to be applied to the pipe in its original condition, before it is expanded by heating, but the error so involved is of the second order and will be here neglected.

The expansion by heating depends on the coefficient of linear expansion for 1 degree Fahrenheit, c, and the rise in temperature in degrees Fahrenheit, T. Every element of the pipe ds expands by an amount $cT\,ds$ in direction of the tangent to the curve at that point and causes every point on SA including A to move by the same amount parallel with that tangent, which forms an angle ψ with OX. Thus the total displacement of A relative to O in direction of OX and OY will be respectively:

$$cT\int_0^A \cos\psi\, ds = cT\int_0^A dx = cTL \text{ and } cT\int_0^A \sin\psi\, ds = cT\int_0^A dy = cTH.$$

The same formulas apply to any point, S, substituting S for A and x and y for L and H. If A is situated on the axis of OX, the heat expansion in direction of OY is zero.

Our fundamental assumption is then that the positive displacements produced by heat expansion are annulled by the negative displacements which would be caused by the application of the reaction forces and couples. Hence we have:

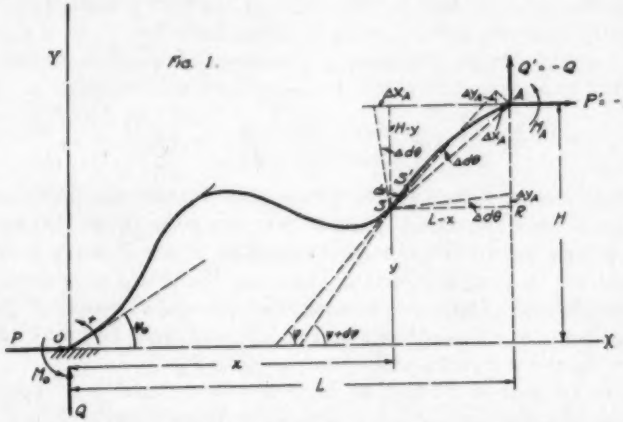
$$cTL+\Delta x_A=0 \quad \text{and} \quad cTH+\Delta y_A=0.$$

For briefness we write $cT=\mu$ and, in order to distinguish the calculated or fictitious displacements from those actually existing we shall denote the former by an accent. Thus we have:

$$\Delta x_A'=-\mu L \quad \text{and} \quad \Delta y_A'=-\mu H.$$

The actual displacements at A are $\Delta x_A=0$ and $\Delta y_A=0$, except when the A-end is free in the X or Y direction, but all other points of the pipe will suffer displacements which at any point S

are here denoted by Δx_S and Δy_S or simply by Δx and Δy. It is necessary in many cases to calculate the latter in order to determine the proper design of the supports at points intermediate between the anchorages.



Consider now Fig. 1 in more detail. The curve OSA represents the axis of a pipe of arbitrary form. O is taken as origin and the coördinate axes are placed in an arbitary position in order to make the treatment as general as possible. In many cases it is convenient to place the coördinate axes so that one of them is parallel to or identical with a line of symmetry in the bend, or, if the pipe is "fixed" at one end to place one of the axes along the tangent at that end. If only one end of a pipe is fixed it is always best to place the origin at that end.

For the sake of briefness the X and Y directions are often referred to as horizontal and vertical respectively, although, actually, since gravity is here neglected, the pipe may be in any plane.

By the method of deflection now to be described we are able to find the displacements of any point on the pipe axis and the rotation at that point, but the displacements as well as the rotation are relative to O and the tangent at O as if this tangent were a fixed line of reference, and this is true whether the pipe is fixed or hinged at O. In other words, for a pipe which is under

the action of certain forces and a couple at the free end A, and provided we disregard heat expansion, the deflections obtained by the formulas now to be given are directly applicable and correct only under the assumption that the pipe is actually fixed at O.

If the pipe is hinged at the origin, we must in general apply a correction to the deflections as explained hereafter.

If we take the heat expansion into account, the actual displacements at intermediate points are found from the formulas:

$$\Delta x_S = \Delta x_S' + \mu x$$
$$\Delta y_S = \Delta y_S' + \mu y.$$

Applying then the method of deflection, assume that the O-end is fixed, while the A-end is acted upon by a given couple M_A and by a given force which has the components $P' = -P$ and $Q' = -Q$ along the X and Y directions, but that the A-end is otherwise unrestrained: Let us try to determine the displacement of the A-end caused by said forces and couple, and let S be any point $(x\ y)$ on the axis of the pipe.

The tangent at S forms an angle ψ with OX and the tangent at an adjacent point S' at a distance ds from S forms an angle $\psi + d\psi$ with OX.

As indicated in the figure, we reckon the rotation to be positive in counter-clockwise direction.

The part $A\,S$ of the pipe exerts at S a couple, M_{AS} which causes the element $S\,S'$ to deflect through an angle $\Delta d\psi$. As explained above [formula (2)]:

$$\Delta d\psi = \frac{KM_{AS}}{EI}. \tag{10}$$

The element $S\,S'$, in rotating, carries with it the part $S\,A$ and the point A suffers displacements which can be determined by regarding the figure $S\,S'\,A\,R$ as rigid and rotating through an angle $\Delta d\psi$ in positive direction. The displacements of A are seen from the figure to be:

in direction of OX: $\quad -\Delta d\psi (H-y)$
in direction of OY: $\quad \Delta d\psi\ (L-x)$.

Since every element of the pipe, excepting that at the origin

O, deflects through an angle $\Delta d\psi$ and each one causes displacements of A in direction of OX and OY, we find the total displacements of the A-end to be:

$$\left.\begin{aligned} \Delta\psi_A &= \int_0^A \frac{KM_{AS}}{EI}\,ds \\ \Delta x_A &= -\int_0^A (H-y)\,\frac{KM_{AS}}{EI}\,ds \\ \Delta y_A &= \int_0^A (L-x)\,\frac{KM_{AS}}{EI}\,ds \end{aligned}\right\}$$

Evidently the same formulas would apply to the rotation and displacements at any other point such as S, simply changing the upper limit of the integration to S, and substituting x_S and y_S for L and H.

The moment M_{AS} might be expressed in terms of the forces and couple acting at A:

$$M_{AS}=M_A+P(H-y)-Q(L-x)$$

but evidently M_{AS} is counterbalanced by an exactly equal and opposite couple, exerted by the part of the pipe $O\,S$ to the left of S and this couple, which we denote by M_{OS} can be expressed more simply than M_{AS}:

$$M_{OS}=M_O+Py-Qx.$$

It is therefore more convenient to write:

$$M_{AS}=-(M_O+Py-Qx) \tag{12}$$

and to substitute this expression in equations (11). Applying accents to the deflections in order to indicate, as explained above, that they are generally fictitious in the problem of pipe bends, we get:

$$\left.\begin{aligned} \Delta\psi_A' &= -\int_0^A \frac{K}{EI}\,(M_O+Py-Qx)ds \\ \Delta x_A' &= \int_0^A \frac{K(H-y)}{EI}\,(M_O+Py-Qx)ds \\ \Delta y_A' &= -\int_0^A \frac{K(L-x)}{EI}\,(M_O+Py-Qx)ds \end{aligned}\right\}. \tag{13}$$

These are the most general and fundamental equations. They apply to all plane bends and, in fact, to all plane curved bars whatever may be their form. They give the displacements and the rotation at A relative to O and relative to the tangent at O regarded as fixed when the A-end is free and under the action of given forces and a given couple. Even if the O-end is not actually fixed in direction, the formulas give the same result as if it were fixed.

It is here to be noted that if we apply formulas (13) first to one end and then to the other end of a given bar or pipe which is in equilibrium under a given system of forces, we shall not find the displacements of the free end to be the same. This may at first sight seem paradoxical, since the same forces and couples are acting at the ends in either case, since the angular deflection of each element must be the same, and since the integration extends between the same limits. It is due to the fact that it is tacitly assumed in these formulas that the tangent at the origin is fixed in either case, so that if, for instance, the changes of curvature under the given system of forces are much greater near one end than the other the deflections must differ widely in the two cases.[4]

Hence these formulas must be applied with a certain caution. When the end at the origin is actually fixed, the formulas give the correct deflection of A if that end is free, and this is true whether we place one of the coördinate axes along the tangent at the origin or not. If we impose certain conditions of restraint on the A-end, we must adjust the forces and the couples acting at O and A accordingly, and this is what we have to do in pipe bends. If both ends are hinged, and we place the origin at one end, the formulas will give us fictitious displacements for the A-end, and, as explained in the next section, it is necessary in that case, if we want to know the vertical displacement at intermediate points, to bring A back to the line $O\ A$ by rotating the whole pipe, as if it were rigid, about O.

We are now ready to apply the formulas to a pipe bend which is restrained at the ends and prevented from expanding in a

[4] We may visualize this if we imagine a dog to be wagged by the tail instead of the tail being wagged by the dog.

certain direction by anchorages when a rise in temperature takes place. In that case the forces P, Q and the couple M_A appear as unknown reactions and all we have to do is to substitute the known components of the heat expansion, μL and μH, with negative sign instead of $\Delta x_A'$ and $\Delta y_A'$ in the last two of equations (13). We obtain thus the following relations, which are applicable in practically all cases of plane bends subject to heat expansion and with immovable anchorages:

$$\left.\begin{aligned} EI\Delta\psi' &= -\int_0^A K(M_O+Py-Qx)ds \\ EI\mu L &= -\int_0^A K(H-y)(M_O+Py-Qx)ds \\ EI\mu H &= \int_0^A K(L-x)(M_O+Py-Qx)ds \end{aligned}\right\}. \quad (13')$$

In these formulas, then, μL and μH are simply the numerical values of the heat expansion.

At any point S on the pipe, the *actual* displacements are found, in accordance with (9) from the equation:

$$\left.\begin{aligned} \Delta x_S &= \Delta x_S' + \mu x = \int_0^S \frac{K(y_S-y)}{EI}(M_O+Py-Qx)ds + \mu x_S \\ \Delta y_S &= \Delta y_S' + \mu y = -\int_0^S \frac{K(x_S-x)}{EI}(M_O+Py-Qx)ds + \mu y_S \end{aligned}\right\}. \quad (13'')$$

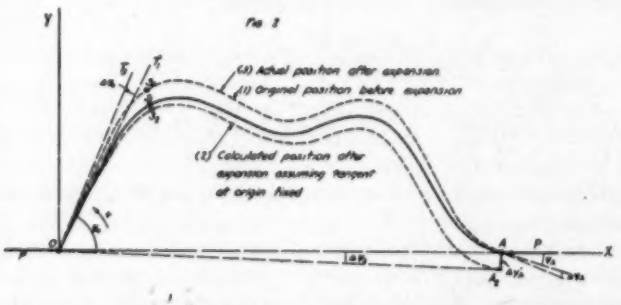
When S is moved to A we have according to (13') $\Delta x_S = \Delta x_A = 0$ and $\Delta y_S = \Delta y_A = 0$.

The boundary conditions always furnish at least one of the unknown quantities. If both ends are fixed $\Delta\psi_O = \Delta\psi_A = 0$. If both ends are hinged $M_O = M_A = 0$ and also $Q = 0$. If one end is fixed, the other hinged, it is simplest to place the origin at the fixed end, and then $M_A = 0$. If the pipe has an axis of symmetry both as regards form and mode of anchorage, the solution is simplified. For instance, if the pipe is symmetrical with respect to a straight line normal to OA and midway between O and A, and if both ends are either fixed or hinged, there can be no reaction forces normal to OA. Hence, if we place OX along or parallel with

OA, the Q-forces will be zero, and it is sufficient to deal with one side of the bend. This is not the case if one end is fixed and the other hinged, because then the pipe will not deform symmetrically.

In practice we meet a variety of plane pipe bends, but they differ only in form and in the terminal conditions. They can all be dealt with by means of equations (13) and (13′) but the mode of attack is not always the same, and by a proper location of origin and axes of coördinates much labor can often be saved. The location of the ends and their mode of fixation usually govern the treatment; but while in some cases these features allow simplifications, they may in others introduce complications, which may easily lead to errors. In order to illustrate these points we shall now make a special study of some of the more important or frequently occurring types of pipe-bends, and show how the fundamental equations can be simplified in many cases.

3. *Simple Plane Bend. Hinged at the Ends.* Consider a curved pipe, represented in Fig. 2 by its axis OS_1A, denoted as position (1). S' is any point on (1) .



The pipe is unsymmetrical in form and, being hinged at both ends, the reaction couples M_O and M_A are equal to zero. The length of the pipe in direction of OX is $OA = L$. The coördinate system is placed, as shown in the figure, so that OX coincides with OA. Since $M_O = M_A = 0$, the resultant reaction forces at O and A must act along OA and it follows that the Q-forces or reactions normal to OX must be zero. Thus, by placing OX through

the end points, we are left with only one unknown reaction, P, and have

$$M_{OS} = Py.$$

Substituting in the first of equations (13′), we find:

$$\Delta\psi_A' = -\frac{P}{EI}\int_0^A Ky\,ds \tag{14}$$

which represents the total rotation of the tangent at A relative to that at O, and is the rotation at A_2 when the tangent at O, denoted in Fig. 2 by OT_1, is regarded as fixed.

From the second of equations (13′) we find, since $H=0$:

$$P = \frac{EI\mu L}{\int_0^A Ky^2 ds}. \tag{15}$$

But the third of equations (13′) does not apply, since $H=0$ and therefore $\mu H=0$, while the integral on the right hand side of the equation is not equal to zero. In fact, this integral according to (13) gives the vertical displacement $\Delta y_A'$ and since P is now determined by (15), we cannot make this displacement vanish. According to (13) A should go to A_2, so that $AA_2=\Delta y_A'$. Apparently the pipe will take up position (2), following the curve AS_2A_2. Yet we know from the physical conditions of the problem that the actual displacement Δy_A must be equal to zero. This discrepancy is explained by the fact that the displacement $\Delta y_A'$ is fictitious, resulting from the artificial assumption that the tangent OT_1 is fixed. Actually the curve AS_2A_2 and therewith its tangent at the origin will take up such a position that A_2 falls in A. We may therefore imagine the whole curve (2) to swing up, as if rigid, until A_2 coincides with A. Then the curve takes up position (3), OS_3A, which represents the actual final position of the pipe after heat expansion has taken place.

Equation (15), which gives the reaction P, is ordinarily the only equation of practical importance in this problem. It is, however, in some cases of interest to know the vertical displacement at intermediate points of the pipe, resulting from the com-

bined effects of heat expansion and deformation, and we shall now show how this can be found.

In swinging the curve from position (2) to position (3) the tangent OT_1 goes to OT_3, rotating in positive direction through an angle, which we call $\Delta\psi_O$, while the tangent at A after rotation forms an angle $\Delta\psi_A$ with its original direction. Evidently the sum of these angles is equal to the total rotation:

$$\Delta\psi_A' = \Delta\psi_A - \Delta\psi_O$$

where $\Delta\psi_A'$ and $\Delta\psi_A$ are both negative, while $\Delta\psi_O$ is positive.

It is also clear that the angle between OA and OA_3 denoted by $\Delta\psi_3$ in the figure is equal to $\Delta\psi_O$, and hence:

$$\Delta\psi_3 = -\Delta\psi_O = \frac{\Delta y_A'}{L}.$$

From the third of equations (13) we find:

$$\Delta y_A' = -\frac{P}{EI}\int_0^A K(L-x)y\,ds \tag{16}$$

$$\therefore \quad \Delta\psi_O = \frac{P}{EIL}\int_0^A K(L-x)y\,ds. \tag{17}$$

The actual vertical displacement at any point S can now be found as the algebraic sum of three quantities, *viz.*, the fictitious displacement $\Delta y'_S$; the displacement produced by swinging the curve from position (2) to position (3) through an angle $\Delta\psi_O$; and the displacement directly due to heat expansion:

$$\therefore \quad \Delta y = -\frac{P}{EI}\int_0^S K(x_S-x)y\,ds + x_S\Delta\psi_O + \mu y. \tag{18}$$

In this case as in all the following, when compressive forces P act at the ends of a pipe, there is at any point S an axial compressive force $P\cos\psi$ and a shear force $P\sin\psi$, where ψ is the angle which the tangent at S makes with OX. A similar remark applies to Q. Usually the displacements and even the stresses due to these forces can be neglected except in experimental work.

When P is found from (15), Δy can be found from (18). The same treatment is applicable in all cases where both ends of a

pipe can be considered as hinged. In illustration of the method we shall here discuss a few simple cases, which can be solved algebraically and which permit of certain generalizations.

3*a*. *Circular Quarter-Bend with Hinged Ends.* The force P is found from (15), in which K can be placed outside the integral sign as it is constant. (See Fig. 3.) We can here evaluate the integral algebraically and find:

$$\int_0^A y^2 ds = \frac{(\pi-3)R^3}{2}$$

$$\therefore \quad P = \frac{2EI\mu L}{(\pi-3)KR^3}. \tag{19}$$

At any point the bending moment is $M = Py$ and the maximum bending moment is:

$$M_{\max} = \frac{\sqrt{2}-1}{\sqrt{2}} PR. \tag{20}$$

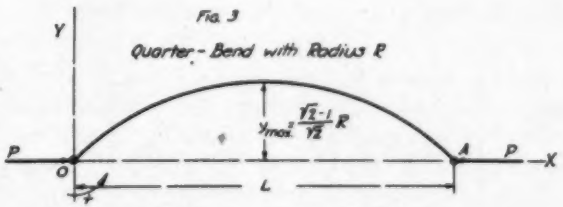


Fig. 3
Quarter-Bend with Radius R

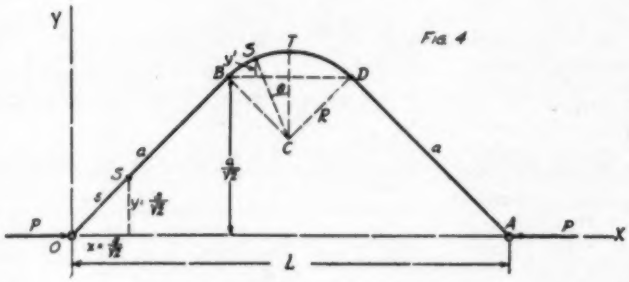
3*b*. *Circular Quarter-Bend, Hinged at the Ends, with Two Straight Branches of Equal Length.* The branches are generally tangents to the ends of the quadrant and are often referred to as 'tangents." In many cases the branches are very long relative to the radius of the quadrant and often they are of unequal length. We shall here consider the case where the branches are of equal length, a.

In Fig. 4 the top of the bend is denoted by T, and $OB = AD = a$. The factor K is unity for the straight branches but has a certain constant value greater than one in the bend.

The reaction P is again found from (15), but the evaluation of the integral is a little more laborious.

In way of the bend $y=y'+\frac{a}{\sqrt{2}}$, where y' is the ordinate referred to the chord BD as abscissa and is given by

$$y'=R\left(\cos\theta-\frac{1}{\sqrt{2}}\right).$$



Outside the bend $y=\frac{s}{\sqrt{2}}$, where s is reckoned from 0.

$$\int_0^A Ky^2ds=2\int_0^B y^2ds+K\int_B^D\left(y'+\frac{a}{\sqrt{2}}\right)^2ds$$

$$=2\int_0^B y^2ds+K\int_B^D\left(y'^2+\frac{a^2}{2}+\frac{2ay'}{\sqrt{2}}\right)ds.$$

$$\int_0^B y^2ds=\int_0^B\frac{s^2}{2}ds=\frac{a^3}{6},\quad \int_B^D y'ds=\frac{4-\pi}{2\sqrt{2}}R^2,$$

$$\int_B^D y'^2ds=\frac{\pi-3}{2}R^2,\quad \int_B^D\frac{a^2}{2}ds=\frac{\pi a^2R}{4}.$$

$$\therefore\quad \int_0^A Ky^2ds=\frac{a^3}{3}+\frac{\pi a^2}{4}KR+\frac{(4-\pi)a}{2}KR^2+\frac{\pi-3}{2}KR^3.$$

$$P=\frac{EI\mu L}{\int_0^A Ky^2ds}=\frac{12EI\mu L}{4a^3+KR[3\pi a^2+6(4-\pi)aR+6(\pi-3)R^2]}. \quad (21)$$

The maximum bending moment is found at T:

$$M_{max} = [a + R(\sqrt{2} - 1)] \frac{P}{\sqrt{2}}. \tag{22}$$

When the branches are long relative to the radius of the bends these formulas can be simplified with sufficient approximation.

When $a > 2R$ we may omit the last term inside brackets in the denominator of (21) and have thus:

$$M_{max} = \frac{12EI\mu L[a + R(\sqrt{2}-1)]}{\sqrt{2}[4a^3 + 3aKR[\pi a + 2(4-\pi)R]]}. \tag{22'}$$

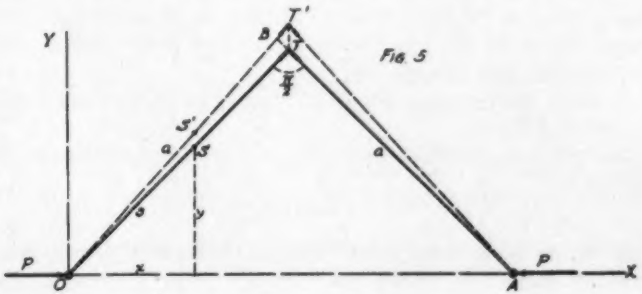
When $a = 2R$ and with $K = 5$, the error of this approximation is about 1.5 per cent.

When $a > 15\ R$ we may omit both the middle and the last term inside brackets in the denominator of (21) and get then the simple expression:

$$M_{max} = \frac{12EI\mu L[a + R(\sqrt{2} - 1)]}{a^2\sqrt{2}(4a + 3\pi KR)}. \tag{22''}$$

When $a = 15R$ and with $K = 5$ the error is again about 1.5 per cent. Ordinarily K is less than 5 and the error will then be smaller in either case.

3*c*. *Cast Right-Angled Knee with Two Long Straight Branches of Equal Length. Hinged at Ends.* The angle at T is assumed to remain a right angle, and the branches, being hinged at O and A take a slight curvature, which is greatest at the top. The



angular deflection of the branches is greatest at O and A. The top point T moves to T'. The displacement $\Delta x_A' = -\mu L = -\mu a\sqrt{2}$, assuming $OT = a$. See Fig. 5. We have $K = 1$ everywhere, and I is uniform except in the knee where it is much greater than elsewhere. We disregard this irregularity and form the integral in the denominator of (15) as if I was the same throughout.

From (15):

$$P = \frac{EI\mu L}{\int_0^A y^2 ds}.$$

$$y = \frac{s}{\sqrt{2}}, \quad \int_0^A y^2 ds = 2\int_0^a \frac{s^2}{2}\, ds = \frac{a^3}{3} \quad \therefore\ P = \frac{3\mu EI\sqrt{2}}{a^2}. \tag{23}$$

$$M_{\max} = \frac{a}{\sqrt{2}} P = \frac{3\mu EI}{a}. \tag{24}$$

It is seen that the maximum bending moment corresponding to a given expansion is inversely proportional to the length of the branches, and this must be true approximately also if an ordinary elastic bend is fitted at T instead of the knee, provided the branches are long relative to the radius of the bend.

On the other hand, when the branches are long, TT' is considerable and has to be provided for in the design of the supports. Referring to Fig. 5, assume the branches OT' and AT' to remain straight. Drop TB normal to OT', then $BT' = \mu a$ and $TT' = BT'\sqrt{2} = \mu a\sqrt{2}$, but this is equal to $-\Delta x_A'$, so that the actual displacement at the top is equal to the heat expansion in the horizontal direction. The same result would have been found by the more complicated formula (18).

Actually the branches are slightly curved and form parabolas of the third degree.

Let the ratio between the length, a, and the diameter of the pipe be m:

$$a = md;$$

then the maximum longitudinal bending stress, which occurs near the knee is:

$$p = \frac{M_{max} d}{2I} = \frac{3\mu E d}{2a} = \frac{3\mu E}{2m}; \tag{25}$$

that is, the stress is inversely proportional to m.

If for instance the coefficient of expansion $c = 8 \times 10^{-6}$, and the range of temperature is 760° F., we have

$$\mu = 8 \times 10^{-6} \times 760 = 6080 \times 10^{-6}.$$

Suppose $m = 100$ and let $E = 30 \times 10^6$ lbs./sq. in.

$$\therefore \quad p = \frac{3 \times 6080 \times 30}{200} = 2736 \text{ lbs./sq. in.}$$

This stress would occur, for instance, in a 12-inch pipe with two branches each 100 ft. long, but even if the branches were as short as 25 ft., the stress would be only about 11,000 lbs./sq. in. The displacement of T is: $TT' = \mu a \sqrt{2}$, which for $a = 100$ ft. = 1200 in. would make $TT' = 6080 \times 1200 \times 1.414 \times 10^{-6} = 10.''3$.

The right angled bend with long branches has the advantage of small stresses, but on the other hand it requires considerable extra length of piping, which will hardly be justified unless local conditions make it necessary or desirable to adopt this type of bend. It must be noted also that it requires intermediate supports especially designed to permit displacements in directions oblique to the pipe.

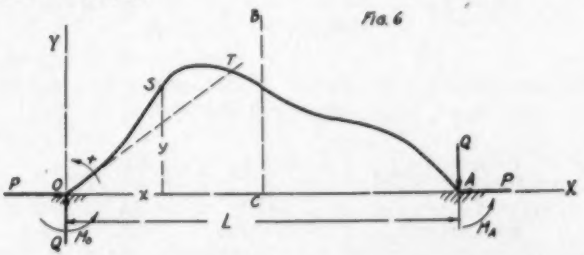


Fig. 6

4. *Simple Plane Bend. Fixed at the Ends.* We place again OX along OA, because in cases of symmetry it gives the simplest solution.

If the pipe, as here assumed, is unsymmetrical, there will be a tendency for it to rotate when it expands by heating. This is

counteracted by the couples M_O and M_A and by vertical forces Q at O and $-Q$ at A. See Fig. 6. It will be shown below that only when the pipe is symmetrical with respect to the middle ordinate, CB, will the Q forces be equal to zero. Taking moments about any point, S, we have:

$$M_{AS} = -M_{OS} = -(M_O + Py - Qx). \tag{26}$$

$$\therefore \quad M_A = -(M_O - QL). \tag{27}$$

Since the tangents at both ends are fixed in direction we have the total angular deflection $\Delta\psi_A$ equal to zero, whence from the first of equations (13′):

$$\int_0^A K(M_O + Py - Qx)ds = 0. \tag{28}$$

The second of equations (13′), when putting $H = 0$, gives:

$$EI\mu L = \int_0^A Ky(M_O + Py - Qx)ds. \tag{29}$$

In this case the fictitious displacement $\Delta y_A'$ as determined by the third of equations (13) must be zero, since $\mu H = 0$, and the fixity of the tangents at the ends precludes a rotation of the pipe as a whole.

With due regard to (28) we find:

$$\int Kx(M_O + Py - Qx)ds = 0. \tag{30}$$

We now introduce the symbol A with appropriate suffixes to denote the various integrals between the limits O to A:

$$\int Kds = A_S, \qquad \int Kxds = A_x, \qquad \int Kyds = A_y$$

$$\int Kxyds = A_{xy}, \qquad \int Kx^2ds = A_{x^2}, \qquad \int Ky^2ds = A_{y^2}$$

and substituting in the foregoing equations we have:

$$A_S M_O + A_y P - A_x Q = 0 \tag{28′}$$

$$A_y M_O + A_{y^2} P - A_{xy} Q = EI\mu L \tag{29′}$$

$$A_x M_O + A_{xy} P - A_{x^2} Q = 0. \tag{30′}$$

From these equations the three unknown quantities P, Q, and M_O can be found readily by the methods of determinants:

$$\Delta M_O = \begin{vmatrix} 0 & A_y & -A_x \\ EI\mu L & A_{y^2} & -A_{xy} \\ 0 & A_{xy} & -A_{x^2} \end{vmatrix}, \tag{31}$$

where

$$\Delta = \begin{vmatrix} A_S & A_y & -A_x \\ A_y & A_{y^2} & -A_{xy} \\ A_x & A_{xy} & -A_{x^2} \end{vmatrix};$$

$$\Delta P = \begin{vmatrix} A_S & 0 & -A_x \\ A_y & EI\mu L & -A_{xy} \\ A_x & 0 & -A_{x^2} \end{vmatrix}; \tag{32}$$

$$\Delta Q = \begin{vmatrix} A_S & A_y & 0 \\ A_y & A_{y^2} & EI\mu L \\ A_x & A_{xy} & 0 \end{vmatrix}. \tag{33}$$

We write out the complete solution for Q:

$$Q = \frac{A_S A_{xy} EI\mu L - A_x A_y EI\mu L}{\Delta} =$$

$$\frac{(A_S A_{xy} - A_x A_y) EI\mu L}{A_S(\overline{A_{xy}}^2 - A_{x^2}A_{y^2}) - A_y(A_x A_{xy} - A_y A_{x^2}) + A_x(A_x A_{y^2} - A_y A_{xy})}, \tag{33'}$$

from which it is seen that $Q=0$ when:

$$\frac{A_S}{A_x} = \frac{A_y}{A_{xy}}.$$

It can easily be shown that this relation holds only when the bend is symmetrical about the middle ordinate, in which case:

$$\frac{\int Kds}{\int Kxds} = \frac{\int Kyds}{\int Kxyds} = \frac{2}{L}, \tag{34}$$

the integrals being taken between the limits O and A.

4*a*. *Circular Quarter-Bend with Fixed Ends.* This case is represented in Fig. 3, except that the ends are here fixed instead of being hinged.

R and K are constants, $Q=0$, and we have two equations for the determination of $M_O=M_A$ and P.

From (28):

$$M_O\int_0^A ds+P\int_0^A yds=0 \text{ or } M_OA_S+PA_y=0 \Big\}. \qquad (28'')$$

From (29):

$$\left.\begin{aligned} M_OK\int_0^A yds+PK\int_0^A y^2ds&=EI\mu L \\ \text{or}\qquad M_OA_y+PA_{y^2}&=EI\mu L \end{aligned}\right\}. \qquad (29'')$$

We might also use (30):

$$M_O\int_0^A xds+P\int_0^A xyds=0, \qquad (30'')$$

but from symmetry (see (34)), this is identical with (28''). As already found in cases (3*a*) and (3*b*):

$$\int_0^A ds=\frac{\pi R}{2},\quad \int_0^A yds=\frac{(4-\pi)}{2\sqrt{2}}R^2,\quad \int_0^A y^2ds=\frac{\pi-3}{2}R^3,$$

and we find from (28'') and (29'')

$$P=\frac{4\pi EI\mu L}{(\pi^2+2\pi-16)KR^3}; \qquad (35)$$

$$M_O=-\frac{4(4-\pi)EI\mu L}{\sqrt{2}(\pi^2+2\pi-16)KR^2}. \qquad (36)$$

The maximum bending moment is at the top and is:

$$M_T=-\frac{4\pi(\sqrt{2}-1)EI\mu L}{\sqrt{2}(\pi^2+2\pi-16)KR^2}. \qquad (37)$$

4*b*. *Circular Quarter-Bend, Fixed at the Ends, with Two Straight Branches of Equal Length.* This case is represented by Fig. 4,

if we imagine the ends to be fixed. Again $Q=0$, and in case 3*b*, we have:

$$y=\frac{s}{\sqrt{2}} \text{ for } OB \quad \text{and} \quad y=\frac{a}{\sqrt{2}}+y' \text{ for the bend.}$$

$$y'=R\left(\cos\theta-\frac{1}{\sqrt{2}}\right), \qquad A_S=\int_0^A Kds=2a+\frac{\pi}{2}KR,$$

$$A_y=\int_0^A Kyds-2\int_0^a yds+K\int_B^D\left(\frac{a}{\sqrt{2}}+y'\right)ds$$

$$=\frac{a^2}{\sqrt{2}}+\frac{\pi a}{2\sqrt{2}}KR+\frac{4-\pi}{2\sqrt{2}}KR^2.$$

As found in case 3*b*:

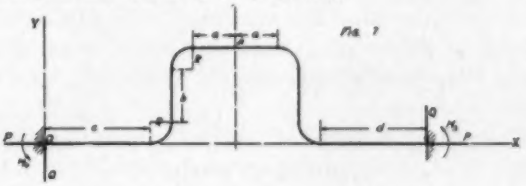
$$A_{yS}=\int_0^A Ky^2ds=\frac{a^3}{3}+\frac{\pi a^2}{4}KR+\frac{(4-\pi)a}{2}KR^2+\frac{\pi-3}{2}KR^3.$$

When the values of the A-coefficients are calculated, M_O and P can be found from (28″) and (29″).

Even if the angle at T is not a right angle, and the bend is not circular, but provided symmetry is preserved, we still have $Q=0$ and the formulas to be used are again (28″) and (29″); but then the evaluation of the A coefficients is generally best done graphically as explained in Section 10.

4*c*. *U-Bend Combined with Straight Parts and Branches. Fixed at Ends.* The pipe consists of two long straight parts *c* and *d*, generally much longer than indicated in Fig 7. An U-shaped bend is inserted consisting of four quadrants drawn with the same radius R and separated by straight parts of lengths *b* and 2*a*.

This pipe does not differ essentially from the general case. The formulas (27), (28), (29), (30) can be used as they stand and a solution obtained from (31), (32), (33).



We shall now assume $c=d$, in which case the lay-out is symmetrical, the vertical reaction $Q=0$ and the solution is much simplified. We need only deal with one-half of the pipe, and can regard the middle point of the top branch as the A-end, which is fixed in direction.

We have then: $\Delta x_A = -\frac{1}{2}\mu L$.

From (28) and (29):

$$M_O\int_0^A Kds + P\int_0^A Kyds = 0;$$

$$\tfrac{1}{2}EI\mu L = M_O\int_0^A Kyds + P\int_0^A Ky^2ds;$$

which give:

$$P = \frac{A_S EI\mu L}{2(A_S A_{y^2} - \overline{A_y}^2)}; \tag{38}$$

$$M_O = -\frac{A_y EI\mu L}{2(A_S A_{y^2} - \overline{A_y}^2)}; \tag{39}$$

$$M_{\max} = \frac{A_S EI\mu L}{2(A_S A_{y^2} - \overline{A_y}^2)} \quad (b+2R). \tag{40}$$

Where A_s, A_y, etc., are the integrals for one-half of the pipe.

The integrals can be evaluated algebraically, but the solution so obtained is cumbrous even in the case of symmetry. The graphical method is believed to be preferable in this case.

5. *Simple Plane Bend — Fixed at One End, Hinged at the Other.* The pipe is fixed at O, both in location and direction, and hinged at A. Under the action of $-P$ applied at A, the pipe will tend to move up or down and a reaction $-Q$ is called forth. At O we have the forces $+P$, $+Q$ and a reaction couple M_O. We make $\Delta y_A' = 0$ because the tangent at O is fixed and A remains in OX. See Fig. 8.

At any point S the bending moment is:

$$M_{OS} = M_O + Py - Qx;$$

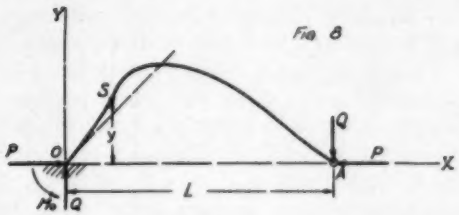


Fig. 8

and since $M_A = 0$ and $y_A = H = 0$, we have, taking moments about A:

$$M_O - QL = 0 \text{ or } Q = \frac{M_O}{L}. \tag{41}$$

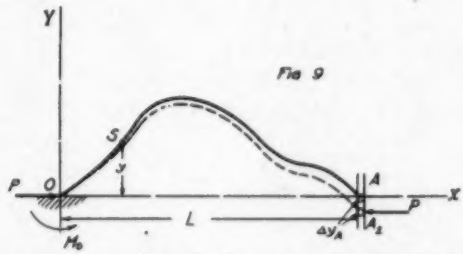


Fig. 9

The general equations (13′) become:

$$\left.\begin{aligned} -EI\Delta\psi_A &= \int_0^A K\left[M_O\left(1-\frac{x}{L}\right)+Py\right]ds \\ EI\mu L &= \int_0^A K\left[M_O y\left(1-\frac{x}{L}\right)+Py^2\right]ds \\ EI\mu H = 0 &= \int_0^A K(L-x)\left[M_O\left(1-\frac{x}{L}\right)+Py\right]ds \end{aligned}\right\} \cdot (42)$$

The last two of these equations give M_O and P, and $\Delta\psi_A$ can then be found from the first if desired. The solution requires the integration of six integrals, but in bends of simple form consisting, for instance, of straight lines and circular arcs this may be readily performed. When M_O is determined, Q can be found from (41).

6. *Simple Plane Bend. Fixed at One End and Free at the Other, Except that it is Prevented from Moving Horizontally.* The pipe, as shown in Fig 9, is fixed at O, while the A-end may be imagined to slide in a vertical guide, which prevents it from moving horizontally, but permits rotation and vertical motion. Hence, when the pipe expands by heating, the only reaction called forth at A is a force $-P$, while $Q=0$ and $M_A=0$. Although $H=0$, the force P will exert a bending moment at the origin, because the A-end will be displaced by an amount Δy_A, which in this case is an actual displacement. The reaction couple at O is:

$$M_O=-P\Delta y_A, \tag{43}$$

and the bending moment at any point is:

$$M_{AS}=-P(y+\Delta y_A)=-M_{OS}. \tag{44}$$

From equations (13) and (13′):

$$\left.\begin{aligned} EI\mu L &= P\int_0^A K(y^2-y\Delta y_A)ds = P(A_{y^2}-\Delta y_A A_y);\\ -EI\Delta y_A &= \\ P\int_0^A K(L-x)(y-\Delta y_A)ds &= P(LA_y-L\Delta y_A A_S-A_{xy}+\Delta y_A A_x) \end{aligned}\right\} \tag{45}$$

From these equations P and Δy_A can be found and hence M_O. It is seen that Δy_A is always negative. For a pipe symmetrical about the middle ordinate it is found that:

$$\Delta y_A=-\frac{A_{xy}P}{A_x P-EI}. \tag{46}$$

7. *Turbine Expansion Bend. 7a. Both Ends Fixed.* This differs from the general case in Section 4, Fig. 6, only in that the pipe is of special form, which renders it convenient to place the coördinate axes parallel with the straight parts which are at right angles to each other. See Fig. 10. Thus the A-end comes to be placed at a height H above O. We have:

From (27):

$$M_{OS}=M_O+Py-Qx;$$

$$\therefore \quad M_A=-(M_O+PH-QL).$$

From (28):

$$\int_0^A K(M_O+Py-Qx)ds=0.$$

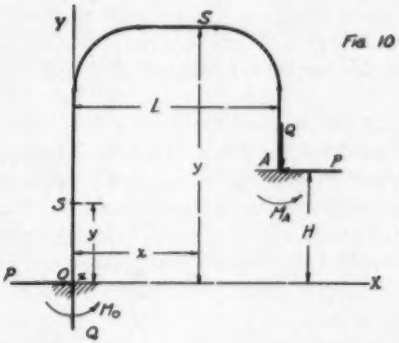


Fig. 10

The horizontal displacement is found from (13′), but in virtue of (28) it reduces to:

$$EI\mu L=\int_0^A Ky(M_O+Py-Qx)ds. \tag{47}$$

Due to the unequal length of the two vertical branches the vertical displacement of A due to heat expansion is not now equal to zero. From (13′) and (28) we find:

$$-EI\mu H=\int_0^A Kx(M_O+Py-Qx)ds. \tag{48}$$

The various integrals are usually best evaluated graphically, after which P, Q, and M_O can be found by the method of determinants from:

$$\left.\begin{aligned} &A_S M_O+A_y P-A_x Q=0\\ &A_y M_O+A_{yy}P-A_{xy}Q=EI\mu L\\ &A_x M_O+A_{xy}P-A_{xx}Q=-EI\mu H \end{aligned}\right\}, \tag{49}$$

where the A coefficients represent the integrals as defined in Section 4.

7*b*. *One End Fixed, the Other Hinged.* The formulas are the same as under Section 5, except that $\Delta y'_A$ is now equal to $-EI\mu H$, which is not equal to zero.

7*c*. *Both Ends Hinged.* The simplest solution is obtained by placing the axis of OX through OA as in Section 3, in which case the formulas (15) to (18) can be applied as they stand.

8. *Offset Bend with Long Branches.* In some cases it may be conveneient for local reasons to offset the pipe as indicated in Figs. 11, 12, and 13, so that it comes to extend on both sides of the line connecting the anchorages O and A. The pipe may be hinged or fixed at the ends, but in no case does it differ essentially from cases 3 and 4 discussed above. There are however certain peculiarities in this type of bend, which make it worth while to consider it separately.

In Fig. 11 such a bend is shown in its most general form, the angles at the bends being unequal and different from a right angle

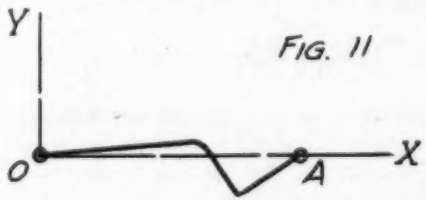


FIG. 11

and the general form unsymmetrical. In Figs. 12 and 13, given below, the bends are both right angled and of the same radius, and the pipe is designed so as to be symmetrical about a middle point C on the line OA. The bends are denoted by D_1DD_2 and E_1EE_2. From symmetry it follows that C is midway between D_2 and E_1.

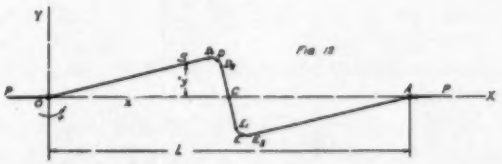
The formulas here given hold good whether the pipe is symmetrically arranged or not, but when the cross branch is nearer one end of the pipe than the other and when C is not in the middle of DE, in fact, in any case of unsymmetry, the whole pipe will generally tend to move out to one side or the other. This is undesirable, since in that case the supports of the pipe must be so

designed as to allow perhaps considerable motion in directions normal to OA. A symmetrical arrangement or one approaching it should therefore be adopted if practicable.

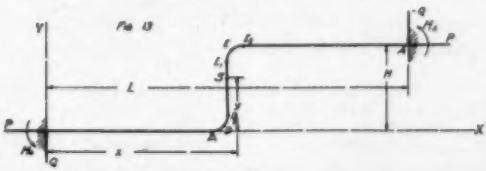
8*a*. *Both Ends Hinged.* It is assumed then that $OC = CA = \frac{L}{2}$; that the bends at D and E are right-angled and circular of the same radius R; and, further, that C is midway between the two bends. See Fig. 12. The only external forces are $+P$ acting at O and $-P$ acting at A, both acting along OA, and it follows that the bending moment at C is zero, so that C is a point of inflection. When the pipe expands due to heating, the line $D_2 E_1$ will rotate, but C will remain in the same position.

As in the single bend, case 3, we have $M_{OS} = Py$ but y may be here positive or negative. The thrust is found from:

$$P = \frac{EI\mu L}{\int_0^A Ky^2 ds}. \tag{15}$$



If we want to find the vertical deflection Δy at D and E, it is in the case of symmetry simply equal to one-half of the expansion of a pipe of length equal to the vertical projection of DE.



8*b*. *Both Ends Fixed.* We place the coördinate axes so that the long branches are parallel to OX and the cross branch parallel to OY. The formulas which are given below would hold also if

the X axis had been placed along OA as in case (a), but the calculation is somewhat simpler with the proposed arrangement. See Fig. 13.

Taking moments about O:

$$M_O+M_A+PH-QL=0.$$

From symmetry M_A must be of the same magnitude and sign as M_O:

$$\therefore \quad 2M_O+PH-QL=0. \tag{50}$$

With the convention that the counter clockwise direction is positive, M_O and M_A will be negative as the figure is drawn.

The three fundamental equations are the same as (49) for the turbine bend, case 7; and we have thus with (50) altogether four equations for the determination of M_O, P, and Q, but the first of (49) is evidently the same as (50) when multiplied by a suitable factor. Hence:

$$\frac{2}{A_S}=\frac{H}{A_y}=\frac{L}{A_x} \tag{51}$$

furnishing two relations between the integrals:

$$A_x=\frac{L}{2}A_S \quad \text{and} \quad A_y=\frac{H}{2}A_S.$$

Thus we obtain from (49) the following three equations:

$$\left.\begin{aligned} &2M_O+HP-LQ=0 \\ &A_yM_O+A_{yy}P-A_{xy}Q=EI\mu L \\ &A_xM_O+A_{xy}P-A_{xx}Q=-EI\mu H \end{aligned}\right\}, \tag{52}$$

which can be readily solved by the method of determinants.

9. *The Lyre-Shaped Bend.* The bend consists of four circular arcs as shown in Fig. 14: the two upper of radius R_2, referred to collectively as (1), the lower of radius R_2 referred to as (2). The angles covered by the arcs are ψ_1 and ψ_2 respectively. The total height of the bend is H and the height of C_1, the center of arc

(1), above the base is d_1:

$$C_1M = d_1 = H - R_1.$$

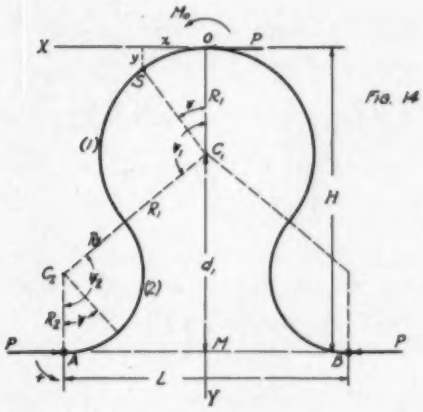


Fig. 14

9a. *Both Ends Hinged.* For convenience we place the origin at O, where the tangent remains in the same direction. The axes OX and OY are placed as shown in the figure. Since the bend and the reactions are symmetrical with respect to OY, we need only deal with one-half of the bend, and consider the part OA.

When the pipe expands, a reaction force $-P$ will be created at A and a force $+P$ at the other end B. At O a stress force $-P$ is exerted by the part BO on OA and there is also a stress couple M_O, but no shear force. No vertical or Q-forces act at the ends.

The displacement $\Delta x_A'$ as determined by the formula (13) is equal to one-half of the total heat expansion in horizontal direction:

$$\Delta x_A' = -\frac{1}{2}\mu L.$$

From (13'):

$$\frac{1}{2}EI\mu L = -\int_0^A K(H-y)(M_O+Py)ds;$$

but since $M_A=0$ we have $M_O=-PH$ and

$$M_{OS}=M_O+Py=-P(H-y)$$

$$\therefore \quad P=\frac{EI\mu L}{2\int_0^A K(H-y)^2ds}. \tag{53}$$

The rotation of the tangent at A, neglecting the rotation caused by heat expansion, is from (13′)

$$\Delta\psi_A'=\frac{P}{EI}\int_0^A K(H-y)ds. \tag{54}$$

The vertical displacement is the algebraic sum of the displacement due to the action of P and that due to heat expansion:

$$\Delta y_A=\Delta y_A'+\mu H=\frac{P}{EI}\int_0^A K\left(\frac{L}{2}-x\right)(H-y)ds+\mu H. \tag{55}$$

Actually the base line AB is fixed and O will move relative to AB by the same amount but in opposite direction.

The integrals may be evaluated algebraically without difficulty, but the expressions are not very simple.

For arc (1):

$$y=R_1(1-\cos\psi) \quad \therefore \quad H-y=d_1+R_1\cos\psi;$$

$$x=R_1\sin\psi \qquad \therefore \quad \frac{L}{2}-x=\frac{L}{2}-R_1\sin\psi.$$

For arc (2):

$$y=H-R_2(1-\cos\psi) \quad \therefore \quad H-y=R_2(1-\cos\psi);$$

$$x=\frac{L}{2}-R_2\sin\psi \qquad \therefore \quad \frac{L}{2}-x=R_2\sin\psi.$$

We find on substitution in (54):

$$\Delta\psi_A'=\frac{P}{EI}\left[K_1R_1(d_1\psi_1+R_1\sin\psi_1)+K_2R_2^2(\psi_2-\sin\psi_2)\right]$$

$$\int_0^A K(H-y)^2ds=K_1R_1(d_1^2\psi_1+\tfrac{1}{2}R_1^2\psi_1+\tfrac{1}{2}R_1^2\sin\psi_1\cos\psi_1+2d_1R_1\sin\psi_1)$$

$$+ K_2R_2^3\left(\frac{3}{2}\psi_2+\frac{1}{2}\sin\psi_2\cos\psi_2-2\sin\psi_2\right). \qquad (54')$$

$$\therefore P=\frac{EI\mu L}{K_1R_1(2d_1^2\psi_1+R_1^2\psi_1+R_1^2\sin\psi_1\cos\psi_1+4d_1R_1\sin\psi_1)+K_2R_2^3(3\psi_2+\sin\psi_2\cos\psi_2-\psi\sin\psi_2)} \qquad (56)$$

In the simple U-bend where

$$\psi_1=\psi_2=\psi;\quad R_1=R_2=R;\quad K_1=K_2=K;\quad d_1=R;$$

we find

$$P=\frac{EI\mu L}{3\pi KR^3}. \qquad (56')$$

9*b*. *Both Ends Fixed.* We refer again to Fig. 14, where in this case terminal couples M_A and $M_B=-M_A$ exist at A and B. Again, by symmetry there can be no Q forces.

From (13′) since $\Delta\psi_A'=0$

$$\int_0^A K(M_O+Py)ds=0 \quad \text{or} \quad M_O=-\frac{A_y}{A_s}P. \qquad (57)$$

From (13′) and in virtue of (57)

$$\tfrac{1}{2}EI\dot{\mu}L=\int_0^A Ky(M_O+Py)ds.$$

Substituting for M_O from (57) and solving for P:

$$P=\frac{EI\mu LA_s}{2(A^2A_{ys}-\overline{A_y}^2)}, \qquad (58)$$

$$\therefore \quad M_O=-\frac{EI\mu LA_y}{2(A_sA_{ys}-\overline{A_y}^2)}; \qquad (59)$$

$$M_A=-(M_O+PH)=\frac{EI\mu L}{2(A_sA_{ys}-\overline{A_y}^2)}(A_y-HA_s). \qquad (60)$$

For the simple U-bend we have:

$$A_s=\pi KR,\quad A_y=\pi KR^2,\quad A_{ys}=\frac{3}{2}\pi KR^3;$$

$$\therefore \quad P=\frac{EI\mu L}{\pi KR^3}; \qquad (58')$$

so that, in order to absorb a given expansion in such a bend with fixed ends, there is required three times as great a thrust as when the ends are hinged.

9c. *One End Fixed, the Other End Hinged.* It is here convenient to place the origin at the fixed end, since the deformation of the bend is not symmetrical. See Fig. 15. The A-end is forced to remain on OX and we must therefore introduce a restraining vertical force Q. The deflection $\Delta\psi_A$ will not be equal to zero. We have:

$$M_{OS}=M_O+Py-Qx,$$

but since

$$M_A=0 \quad \text{and} \quad H=y_A=0,$$

we have

$$M_A=M_O-QL=0.$$

$$\therefore \quad M_O=QL$$

and

$$M_{OS}=M_O\left(1-\frac{x}{L}\right)+Py. \tag{61}$$

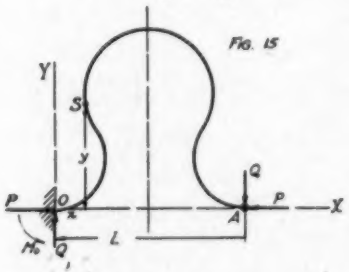


FIG. 15

From (13')

$$\left.\begin{aligned} EI\Delta\psi_A &= -\int_0^A K\left[M_O\left(1-\frac{x}{L}\right)+Py\right]ds \\ EI\mu L &= \int_0^A K\left[M_O y\left(1-\frac{x}{L}\right)+Py^2\right]ds \\ EI\mu H &= \int_0^A K(L-x)\left[M_O\left(1-\frac{x}{L}\right)+Py\right]ds=0 \end{aligned}\right\}. \tag{62}$$

For the simple U-bend we find:

$$P = \frac{EI\mu L}{1.44\pi KR^3}, \tag{63}$$

which is intermediate in value between that for hinged ends and that for fixed ends.

9*d*. *One End Fixed, the Other End Free to Move Vertically.* This is case 6 and the equations are the same as (44) and (45).

10. *Graphical Method of Calculation.* The solution of the fundamental equations (13′) depends on the evaluation of one or more integrals. In many cases, as where the pipe is straight or bent to circular arcs, the integration can be most readily performed algebraically as in 3*a*, 3*b*, 4*a*, 4*b* above, but often the algebraic method is inaccurate or too laborious, and in some cases even quite inapplicable. It is then necessary to resort to graphical integration, which indeed can be applied in all cases.

Consider an integral such as:

$$A_{y^2} = \int_0^A Ky^2 ds.$$

In order to evaluate this integral, calculate the expression Ky^2 under the integral sign for a number of points on the axis of the pipe, assigning to K and y their proper values at each point. The quantities so obtained are plotted as ordinates on the developed girth of the axis of the pipe as abcissae. The area under the curve may now be measured with any integrating instrument and, after multiplication with an appropriate scale factor, it will give the value of the integral. We reproduce here as illustration the calculation for the deflection of *a lyre-shaped pipe bend*, taken from the 1926 paper. The bend was presented to the Institute by the Walworth Manufacturing Company and was tested to a load of 1200 lbs. It was so supported that it could be regarded as hinged at both ends. The deflection, denoted by $2\Delta x_P$ for the whole bend was calculated, as clearly shown in Table IV of that paper here appended, from the formula:

$$2\Delta x_P = \frac{P}{EI} \int K(H-y)^2 ds.$$

TABLE IV.

WALWORTH PIPE BEND

GRAPHICAL COMPUTATION OF $(\Delta X)_1$

POINT	R	(H-y)	$(H-y)^2$	K	$K(H-y)^2$	POINT	R	(H-y)	$(H-y)^2$	K	$K(H-y)^2$
R_0 :O	30·31	94·73	8974	2·0720	18594	R_0 :O					
R1	30·31	85·80	7362	2·0720	15253	L1	30·31	85·80	7362	2·0720	15253
R2	30·31	64·13	4113	2·0720	8522	L2	30·31	64·03	4100	2·0720	8494
R3	30·02	42·35	·1796	2·0904	3755	L3	29·63	42·25	1785	2·1156	3777
R4	30·02	29·50	870	2·0904	1697	L4	29·63	29·33	860	2·1156	1703
R5	29.12	8·03	64·5	2·1500	139	L5	29·63	7·80	60·8	2·1156	129
R6	∞	0	0	1·000	0	L6	∞	0	0	1·000	0
R7 :F	∞	0	0	1·000	0	L7 :F	∞	0	0	1·000	0

$$K = \frac{4\beta^2R^2 + 10\lambda^2}{4\beta^2R^2 + \lambda^2} \qquad \beta = .13225 \quad R = \text{AS TABULATED} \quad \lambda = 3.196$$

HORIZONTAL SCALE 1" = 8" VERTICAL SCALE 1" = 2000 Ω"

1 Ω" AREA UNDER CURVE = 8 × 2000 = 16000 in.^{2}

INTEGRATED AREA = 112 Ω"

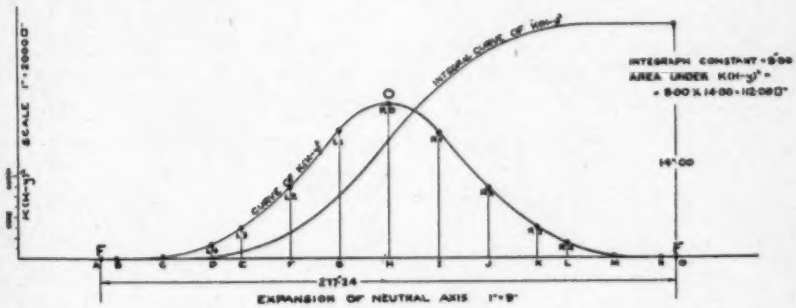
$$2(\Delta X)_1 = \frac{P}{EI} \times 112 \times 16000 = \frac{1200 \times 1{,}793{,}000}{29 \times 10^6 \times 27.176} = 2.7290$$

i.e. 2·73 #

OR $2(\Delta X)_1$ = 2·73 COMPUTED BY ALGEBRAIC METHOD, AND
$2(\Delta X)_1$ = 2·94 OBSERVED

This will be found to be identical with (53) of this paper, bearing in mind that the integration extended over both sides of the bend, because they were not exactly symmetrical. It will be noticed that K varies from point to point due to changes in curvature. The integral was evaluated by integration of a curve for $K(H-y)^2$ shown in Fig. 16 to a reduced scale. The scale of

FIG. 16.



INTEGRATION OF CURVE OF $K(H-y)^2$ FOR GRAPHICAL COMPUTATION OF $(\Delta X)_F$ FOR

THE WALWORTH PIPE BEND

abcissae was 1 in. = 8 in. and the ordinates were plotted to a scale 1 in. = 2,000 in.^{2} The area of this curve was found to be 112.0 sq. in. and as each square inch has a value of $8 \times 2{,}000 = 16{,}000$ in.^{3}, the displacement was

$$2\,\Delta x_F = \frac{P}{EI}\,112.0 \times 16{,}000 = \frac{1200}{29 \times 10^6 \times 27.17} \times 1792 \times 10^3 = 2.73 \text{ in.}$$

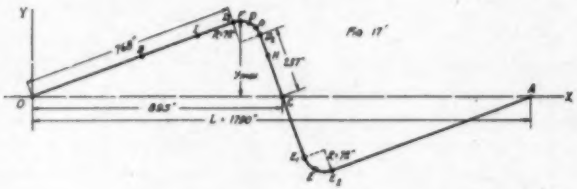
In further illustration of the graphical method we give the calculation for *an offset bend hinged at both ends*, similar to that discussed under 8*a*.

The pipe is of 10 in. nominal diameter and its section has a moment of inertia of 211 in.^{4} The general dimensions are as given in Fig. 17. Since the pipe is symmetrical about C, we need only

make the calculation for one-half of its length. The value of K for the quarter bends at D and E was found to be, $K=1.39$. The coefficient of expansion was taken to be $c=8\times10^{-6}$ and the range of temperature $T=760°$ F. Thus:

$$\mu=8\times10\times760^{-6}=6080\times10^{-6}.$$

Since $L=1790$ in., we find: $\mu L=10.9$ in. for the whole pipe from O to A.



The modulus of elasticity was taken $E=30\times10^6$ lbs./sq. in. The formula to be used is (15):

$$P=\frac{EI\mu L}{\int Ky^2ds}.$$

The following table gives the calculation for the section ODC.

Calculation for $\int ky^2ds$

All Dimensions are in Inches

Point	*s*	*y*	y^2	*K*	Ky^2
0	0	0	0	1	0
B	420	145	21,030	1	21,000
I	630	217	47,090	1	47,100
D_1	768	263	69,170	1	69,200
D_1	768	263	69,170	1.39	96,100
F	795	270	72,900	1.39	101,300
D	825	262	68,640	1.39	95,400
G	853	246	60,510	1.39	84,100
D_2	881	222	49,280	1.39	68,500
D_2	881	222	49,280	1	49,300
H	964	145	21,030	1	21,000
C	1,118	0	0	1	0

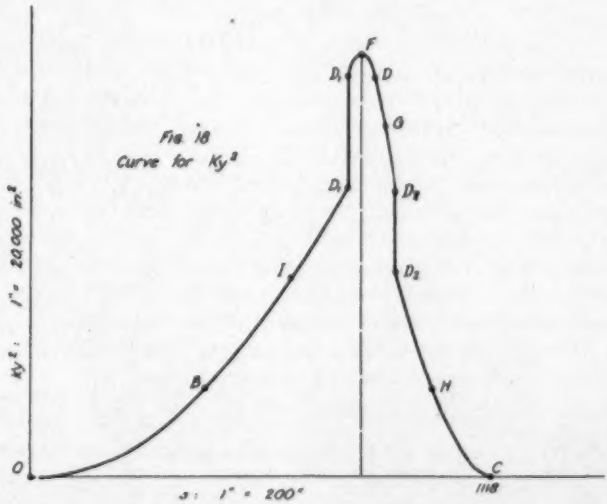
It will be noticed that for the points D_1 and D_2, at the transition between the straight and the curved parts, two values of Ky^2 are given for each of these points. The curve for Ky^2 here takes a vertical rise, providing thus in an approximate manner for the change in curvature.

Fig. 18 shows the curve for Ky^2. The abcissae represent s, the girth of the pipe along the axis, to a scale of $1'' = 200$ in., and the ordinates represent Ky^2, plotted to a scale of $1'' = 20{,}000\ \overline{\text{in.}}^2$ Thus the value of one square inch of the area of the curve is $4 \times 10^6\ \overline{\text{in.}}^3$ This area, which represents the value of the integral for one-half of the pipe, was found to be 8.14 sq. in.

Hence the value of the integral for the whole pipe is:

$$\int_0^A Ky^2 ds = 2 \times 4 \times 10^6 \times 8.14 = 65.1 \times 10^6\ \overline{\text{in.}}^3$$

$$\therefore \quad P = \frac{30 \times 10^6 \times 211 \times 10.9}{65.1 \times 10^6} = 1060 \text{ lbs.}$$



and the maximum bending moment, which occurs at F, is

$$M_{\max} = 270 \times 1060 = 286{,}000 \text{ in. lbs.}$$

All other integrals can be dealt with in the same manner without difficulty, but it must be admitted that in long and involved pipe lines such as occur in many modern plants, the work may become very lengthy. Such pipe lines must be divided by anchorages into a great number of sections, each of which may constitute a separate problem. For this reason it is highly desirable that methods should be devised by which results can be obtained more readily and quickly.

11. *Other Methods of Solution.* Several methods for shortening the work are available, but none of them have yet been fully developed.

a. A summary of formulas for bends of simple geometrical forms such as exemplified in Sections 3 and 4. Several attempts have been made in this direction,[1] some of them very elaborate and complete in their respective fields.

b. Preparation of curves and tables covering a wider field of bends.

c. Model experiments. Mr. S. S. Sanford[2] has made experiments on models of pipe bends of simple shape made up of straight pieces of tubing, and the results were in close accord with the calculated results. He has given formulas and rules for converting the forces, moments, and deflections obtained in model tests into the corresponding values for full-size installations. Such experiments seem very promising, and may perhaps offer the only practicable solution of a very difficult problem, *viz.*, the design of pipe bends in three dimensions, which are quite common and often unavoidable. The author has attempted to deal mathematically with such bends, but due to the fact that both torsion and bending are involved, the calculations become very lengthy and difficult, and apparently unsuitable for practical purposes.

[1] S. Crocker and S. S. Sanford: "The Elasticity of Pipe Bends," Am. Soc. Mech. Eng., 1922.
A. M. Wahl: "Stresses and Reactions in Expansion Pipe Bends," American Society of Mechanical Engineers, 1927.
J. R. Finniecome: "The Flexibility of Plain Pipes," The Metropolitan-Vickers Gazette, Jan. to May, 1928.

[2] See discussion of Mr. Wahl's paper referred to above.

FURTHER RESEARCH ON PIPE BENDS

By Prof. William Hovgaard

A. Introduction

This communication may be regarded as the second part of a paper of which that published in the Journal of Mathematics and Physics, October, 1928,[1] was the first part, both being a continuation of that published in this Journal in 1926.[2]

In Chapter B is given a discussion of the stresses and strains in pipe bends. New formulas are developed by which the interaction between the longitudinal and the transverse strains is taken into account.

Chapter C contains a description of tests made under the author's guidance at the Massachusetts Institute of Technology by student officers in the course of Naval Construction. The first and major set of tests was made in the early part of 1928, and a second set in October, 1928.

Chapter D gives an analysis of the tests, comprising curves and tables which facilitate a general survey of the results. It also contains a description and discussion of the Lüders' lines which appeared on the surface of the pipes.

Chapter E gives a discussion of the strength of pipe bends, applying the modern theory on this subject to the tests.

Chapter F gives summary and conclusions.

It is necessary in order to make this paper fairly independent of the 1926 paper to explain briefly the principal points in the latter. This was done also in the first 1928 paper, from which the following is extracted.

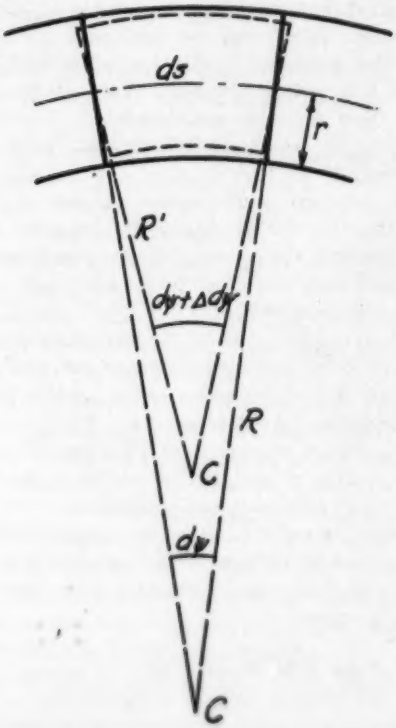
Summary of the 1926 Paper. When a curved pipe is subject to bending, the circular section tends to take a flattened oval form, whereby the stresses in the material farthest from the neutral axis are relieved. The stresses, instead of following the

[1] Deformation of Plane Pipes, Journ. of M. & P. Vol. VII, No. 3, October 1928, hereafter referred to as the "first 1928 paper."

[2] "The Elastic Deformation of Pipe Bends," Journ. of M. & P. Vol. VI, No. 2, November 1926, hereafter referred to as "the 1926 paper."

usual linear law, will be so distributed as to have their maximum values nearer to the neutral axis, and the angular deflection of the bend will be much increased before equilibrium with a given bending moment is established. This was pointed out already in

FIG. 1

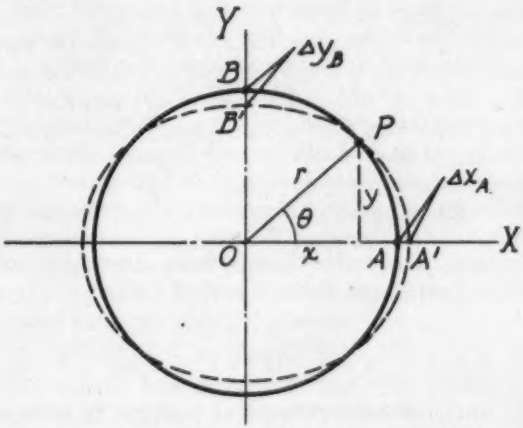


1911 by Dr. v. Karman[3] who developed formulas for the augmented deflections. These formulas remained unconfirmed for a number of years and were by some contested, by others ignored.

[3] Zeitschr. Vereines Deutscher Ingenieure, 1911, Vol. 55, pg. 1889.

In the 1926 paper the author proved the correctness of Karman's formulas by an entirely different method and published the results and an analysis of a series of tests made at the Massachusetts Institute of Technology. This analysis as well as that of tests previously made by Professor Bantlin of Stuttgart, fairly conclusively confirmed the theory.

FIG. 2

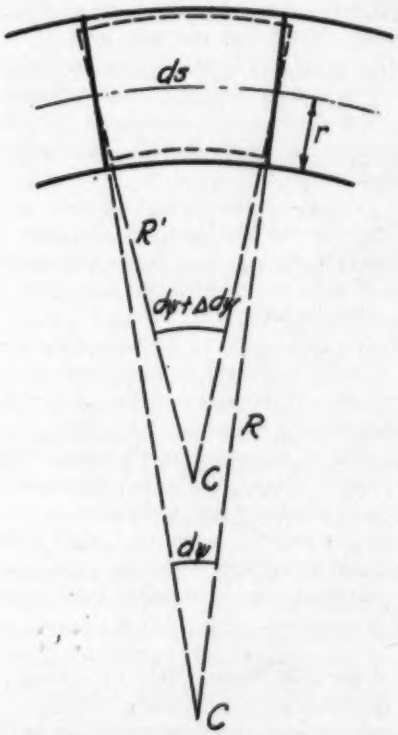


The following notation, which was used in the first paper, will also be used here (see Fig. 1 and 2):

M = bending moment.
R = radius of curvature of axis of pipe.
$2h$ = wall thickness of pipe.
r = radius of the middle surface of the pipe wall.
θ = angle which a radius of the transverse section forms with OX (Fig. 2).
I = moment of inertia of the area of the transverse section of the pipe wall about the neutral axis.
E = modulus of electricity.

usual linear law, will be so distributed as to have their maximum values nearer to the neutral axis, and the angular deflection of the bend will be much increased before equilibrium with a given bending moment is established. This was pointed out already in

FIG. 1

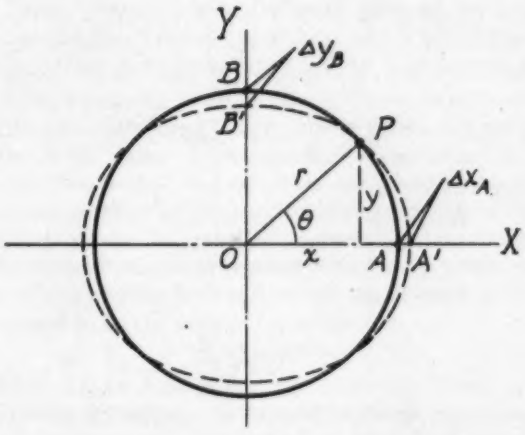


1911 by Dr. v. Karman[3] who developed formulas for the augmented deflections. These formulas remained unconfirmed for a number of years and were by some contested, by others ignored.

[3] Zeitschr. Vereines Deutscher Ingenieure, 1911, Vol. 55, pg. 1889.

In the 1926 paper the author proved the correctness of Karman's formulas by an entirely different method and published the results and an analysis of a series of tests made at the Massachusetts Institute of Technology. This analysis as well as that of tests previously made by Professor Bantlin of Stuttgart, fairly conclusively confirmed the theory.

FIG. 2



The following notation, which was used in the first paper, will also be used here (see Fig. 1 and 2):

M = bending moment.
R = radius of curvature of axis of pipe.
$2h$ = wall thickness of pipe.
r = radius of the middle surface of the pipe wall.
θ = angle which a radius of the transverse section forms with OX (Fig. 2).
I = moment of inertia of the area of the transverse section of the pipe wall about the neutral axis.
E = modulus of electricity.

$d\psi$ = angle of a small arc of the pipe axis comprised between two consecutive transverse sections (Fig. 1).

$\Delta d\psi$ = the change in $d\psi$ consequent on the flexural strain.

s = the length or girth of the axis reckoned from some point of reference to the section under consideration, $ds = Rd\psi$.

According to the ordinary approximate theory we have:

$$\frac{\Delta d\psi}{d\psi} = \frac{MR}{EI} \quad \text{or} \quad \Delta d\psi = \frac{M}{EI} ds \tag{1}$$

which are the formulas commonly used for curved bars. In the 1926 paper it was shown that due to flattening of the section the angular deflection of an elemental piece of a curved pipe was actually

$$\frac{\Delta d\psi}{d\psi} = K\frac{MR}{EI} \quad \text{or} \quad \Delta d\psi = \frac{KM}{EI} ds \tag{2}$$

where $$K = \frac{48h^2R^2 + 10r^4}{48h^2R^2 + r^4} . \tag{3}$$

The value of this coefficient depends on the relative values of h, r and R. Karman put $2hR/r^2 = \lambda$ which makes:

$$K = \frac{12\lambda^2 + 10}{12\lambda^2 + 1} . \tag{4}$$

The factor K, which is greater than one, has the same effect as if the bending moment were increased; it varies from unity in straight pipes to 4 or 5 in pipes bent to a sharp curve. It occurs as a simple factor in the expressions for the longitudinal and transverse stresses p_1 and p_2 and in the displacements of the end of the pipe due to its deformations. It might also be regarded as associated with the moment of inertia, reducing the latter to I/K.

In the tests on pipe bends carried out at the Massachusetts Institute of Technology in 1926, strain measurements of the pipe wall were carried out for the first time and, as explained in that paper, showed a close agreement with the formulas.

It was shown also, both theoretically and experimentally, that the transverse stresses occurring at the ends of the major and minor axes of the deformed transverse section, might under cer-

tain circumstances be much greater than the longitudinal stresses, a fact which had not hitherto been taken into account.

Two bends were tested and one of these, a six-inch lyre-shaped bend, was used again in recent tests, October 1928, which will be described below.

Other Investigations. It is of interest at this point to give a brief account of the work done in this field by certain other investigators since the publication of the 1926 paper.

In a paper[4] presented to the American Society of Mechanical Engineers in 1927, Mr. A. M. Wahl gave formulas, tables and diagrams facilitating the determination of the forces and couples produced at the fixed ends of pipe bends of various standard forms when subject to a rise in temperature. In the analysis he made use of the maximum-shear theory, according to which the strength depends on the value of the maximum shear stress. It follows from this theory that the difference between the algebraically greatest and smallest of the principal stresses, which he called the "equivalent stress," p_e, must not exceed a certain value.[5] The longitudinal and transverse stresses, p_1 and p_2, of which the latter occurs on every radius both as a positive and a negative quantity, are assumed to be the principal stresses:

$$\therefore \quad p_e = p_1 - p_2 \tag{5}$$

Mr. Wahl showed how to find the maximum values of p_e and simplified the calculations by expressing the equivalent maximum stress in terms of the ordinary maximum longitudinal stress, applying to the latter a coefficient a. He wrote:

$$p_e = a \frac{Mr}{I}.$$

The factor a was plotted as ordinates in a diagram on Karman's coefficient λ as abscissae. The equivalent stress shows a maximum at some point not far from the neutral axis and other maxima or extremes at the top and bottom of the section, all of which may be much in excess of the maximum values of the longitudinal and transverse stresses considered apart. Wahl proposed the

[4] "Stresses and Reactions in Expansion Pipe Bends."

[5] This must not be confounded with the "equivalent stress" as defined in this paper, Chapters B and E.

use of the equivalent stress as a criterion for strength, but it is to be noted, first, that the expressions for the longitudinal and transverse stresses used by Wahl are incomplete, since no account is taken in them of the transverse and longitudinal strains respectively, a remark which will be fully explained in the next chapter. Second, as explained in Chapter E, it is doubtful whether the equivalent stress, even when calculated from the complete formulas given below, (20) and (21), can be used as a criterion of strength on account of the local nature of the transverse stresses.

Wahl showed that the steam pressure in a pipe does not reduce the flattening of the pipe section materially.

Wahl made some tests on U-shaped bends of pipes of 1 and 2-in. diameter and found a good correspondence between the observed deflections and those calculated under inclusion of the factor K.

Finally shall be mentioned a series of articles by an English engineer, Mr. J. R. Finniecome, which were published recently.[6] Finniecome analyzed the tests made by Bantlin, by Crocker and Sanford, and also those made by the author in 1926. He made an elaborate analysis of the stresses including a study of the equivalent stress, based on the shear stress theory. He concluded that the actual deflection and stresses agree very closely with the formulas presented by Karman and the author, and recommended that, as proposed by Mr. Wahl, the design of steam piping should be based on the "equivalent stress," $p_e = p_1 - p_2$.

B. The Stresses and the Strains

In the 1926 paper no attempt was made to take account of the interaction between the longitudinal and transverse stresses as determined by Poisson's ratio, because the transverse strain measurements were very uncertain, but in the tests here described a method was devised by which the transverse strains could be obtained about as accurately as the longitudinal strains. Thus it became possible to make a more accurate analysis in which the combined transverse and longitudinal strains and stresses were

[6] "The Flexibility of Plain Pipes," The Metropolitan Vickers Gazette; January to May 1928, afterwards reprinted in the London "Engineer," and in Metropolitan Vickers Circular No. 1440/9-1, July 1928.

used, and we shall now develop the requisite formulas for this purpose.

We assume that a plane stress exists in the wall of the pipe, since there was no radial stress, p_3, in the experiments, but the corresponding strain ϵ_3 is not equal to zero and is taken into account.

It is further assumed that the longitudinal stress p_1, the transverse stress p_2, as well as the radial stress p_3 are principal stresses. This is strictly true only in the section at the top of the bend, where the end forces P, which produce the bending, are normal, so that no shear forces act in the plane of the section. It is approximately true for sections near the top of the bend; but nearer the ends, where the direction of the P forces is more nearly tangential to the sections, the assumption is permissible only when these forces are relatively small. In practice this is generally the case, and even in the tests of the 9½-in. and the 14-in. pipe bends here analyzed, where the forces applied were unusually great, the shear stresses were small. The analysis, moreover, was restricted to sections at or near the top of the bends.

We refer to the notation given in Chapter A and to the 1926 paper. From the Theory of Elasticity we know that the principal stresses p_1, p_2 and p_3 are connected with the principal strains by the formulas:

$$\left.\begin{aligned} p_1 &= 2G\left(\epsilon_1 + \frac{e}{m-2}\right) \\ p_2 &= 2G\left(\epsilon_2 + \frac{e}{m-2}\right) \\ p_3 &= 2G\left(\epsilon_3 + \frac{e}{m-2}\right) \end{aligned}\right\} \qquad (6)$$

where $e = \epsilon_1 + \epsilon_2 + \epsilon_3$ and $G = \dfrac{mE}{2(m+1)}$.

Since the radial stress p_3 is equal to zero, it follows from the last of equations (6) that

$$\epsilon_3 = -\frac{\epsilon_1 + \epsilon_2}{m-1}$$

and substituting in the two first of equations (6), we find:

$$\left.\begin{aligned} p_1 &= \frac{m^2 E}{m^2-1}\left(\epsilon_1+\frac{\epsilon_2}{m}\right) \\ p_2 &= \frac{m^2 E}{m^2-1}\left(\epsilon_2+\frac{\epsilon_1}{m}\right) \end{aligned}\right\} \tag{7}$$

In the 1926 paper it was shown[7] that, with due regard to the flattening of the section, the longitudinal strain can be calculated from the formula:

$$\epsilon_1 = \frac{r}{R}\left(\sin\theta - \frac{6r^4\sin^3\theta}{24h^2R^2+5r^4}\right)\frac{\Delta d\psi}{d\psi}. \tag{8}$$

The transverse strain, in cases where the deflections are small, can be calculated for a point which is at a distance z from the middle surface by means of the formula:

$$\epsilon_2 = z\left(\frac{1}{r'}-\frac{1}{r}\right) \tag{9}$$

where r and r' are the radii of curvature of the transverse section of the middle surface before and after deformation respectively. At the surface of the pipe $z=\pm h$. The term inside brackets is the change in curvature, which, assuming the section to be originally circular, is given approximately by the formula:

$$\frac{1}{r'}-\frac{1}{r} = -\frac{1}{r^2}\left(\Delta r+\frac{d^2(\Delta r)}{d\theta^2}\right) \tag{10}$$

where Δr is the radial displacement. Let Δt be the tangential displacement, then the vertical displacement is:

$$\Delta y = \Delta r \sin\theta + \Delta t \cos\theta \tag{11}$$

and assuming the middle surface to be inextensible we have:

$$\Delta r d\theta + d(\Delta t) = 0 \quad \text{or} \quad \Delta r = -\frac{d(\Delta t)}{d\theta}. \tag{12}$$

Assuming now, as in the 1926 paper,

$$\Delta y = -kr^3\sin^3\theta \tag{13}$$

[7] See formulas 3 and 18 of that paper, putting $y = r\sin\theta$.

$$\dot{M} = \int_0^{2\pi} \int_{-h}^{+h} r p_1 y \, dz \, d\theta.$$

Substituting for p_1 from (7):

$$M = \frac{m^2 r E}{m^2 - 1} \int_0^{2\pi} \int_{-h}^{+h} \left(\epsilon_1 + \frac{\epsilon_2}{m} \right) r \sin \theta \, dz \, d\theta.$$

we find:

$$\Delta t = -\tfrac{1}{2}kr^3 \sin 2\theta$$

from which in conjunction with (12):

$$\Delta r = kr^3 \cos 2\theta. \tag{14}$$

Substituting in (10) we find:

$$\frac{1}{r'} - \frac{1}{r} = 3kr \cos 2\theta \tag{15}$$

where, as given by equations (15) and (18) of the 1926 paper:

$$k = \frac{-\Delta y}{r^3 \sin^3\theta} = \frac{6r^2}{24h^2R^2 + 5r^4} \frac{\Delta d\psi}{d\psi}.$$

Substituting in (15) above and then in (9) we find:

$$\epsilon_2 = 3zkr \cos 2\theta = \frac{18zr^3 \cos 2\theta}{24h^2R^2 + 5r^4} \frac{\Delta d\psi}{d\psi}. \tag{16}$$

The angular deflection $\Delta d\psi / d\psi$ enters both in ϵ_1 and ϵ_2 and is determined as in the 1926 paper by equating the integral of the moments of the longitudinal stresses about the neutral axis to the known bending moment:

$$M = \int_0^{2\pi} 2hr p_1 (y + \Delta y) d\theta.$$

Substituting from (7) above and from (18) in the 1926 paper:

$$M = 2hr \frac{m^2 E}{m^2 - 1} \int_0^{2\pi} \left(\epsilon_1 + \frac{\epsilon_2}{m}\right) \left(r \sin\theta - \frac{6r^5 \sin^3\theta}{24h^2R^2 + 5r^4} \frac{\Delta d\psi}{d\psi}\right) d\theta.$$

Carrying out the integration and bearing in mind that the integral of ϵ_2 vanishes, we obtain by proceeding as in the 1926 paper the same expression for $\Delta d\psi / d\psi$ with the exception that it is multiplied by the factor $(m^2 - 1)/m^2$.

$$\frac{\Delta d\psi}{d\psi} = \frac{m^2 - 1}{m^2} \frac{MR}{EI} \left(\frac{48h^2R^2 + 10r^4}{48h^2R^2 + r^4}\right) = \frac{m^2 - 1}{m^2} \frac{KMR}{EI}. \tag{17}$$

Substituting in (8) and (16), we find:

$$\epsilon_1 = \frac{m^2 - 1}{m^2} \frac{r}{} \frac{KM}{EI} \left(\sin\theta - \frac{6r^4 \sin^3\theta}{24h^2R^2 + 5r^4}\right) \tag{18}$$

$$\epsilon_2 = \frac{m_2 - 1}{m_2} \frac{KMR}{EI} \frac{18zr^3 \cos 2\theta}{24h^2R^2 + 5r^4}. \tag{19}$$

Substituting in (7) from (18) and (19) we find for the outer surface, where $z = h$, the principal longitudinal stress:

$$p_1 = \frac{KMr}{I}\left[\sin\theta - \frac{6r^4 \sin^3\theta}{24h^2R^2 + 5r^4} + \frac{18hr^2R\cos 2\theta}{m(24h^2R^2 + 5r^4)}\right] \tag{20}$$

and the principal transverse stress:

$$p_2 = \frac{KMr}{I}\left[\frac{1}{m}\sin\theta - \frac{6r^4 \sin^3\theta}{m(24h^2R^2 + 5r^4)} + \frac{18hr^2R\cos 2\theta}{(24h^2R^2 + 5r^4)}\right]. \tag{21}$$

The stresses p_1 and p_2, as determined from these formulas, are hereafter referred to as the "*calculated stresses.*" It will be noticed that the factor $m^2 - 1/m^2$ does not appear in these expressions.

The stresses can be calculated for every point of a transverse section of a given pipe from (20) and (21), and when plotted in curves the maximum values can be obtained, but we can also find the maximum values by differentiation of p_1 and p_2 with respect to θ, equating the derivatives to zero. For instance, differentiating the expression for p_1 with respect to θ and equating to zero, we find the value of θ for which p_1 is a maximum to be given by:

$$\sin\theta = -\frac{2hR}{mr^2} \pm \sqrt{\frac{4h^2R^2}{m^2r^4} + \frac{24h^2R^2 + 5r^4}{18r^4}}. \tag{20a}$$

This gives one positive and one negative value for θ and hence a maximum tensile stress and a maximum compressive stress for each side of the section. It is seen that the maximum compressive stress occurs at a greater distance from the neutral axis than the maximum tensile stress.

At the middle surface $p_2 = 0$ and the last term inside brackets in the expression for p_1 vanishes since $z = 0$. The longitudinal stress at the middle surface is a maximum when

$$\sin\theta = \pm\sqrt{\frac{24h^2R^2 + 5r^4}{18r^4}} \tag{20b}$$

and the maximum value is:

$$p_{1\max.} = \pm\frac{2}{3}\frac{KM}{rI}\sqrt{\frac{24h^2R^2 + 5r^4}{18}}. \tag{21}$$

Also, since $p_1 = E\epsilon_1$ at the middle surface, we have here

$$\frac{\Delta d\psi}{d\psi} = \frac{KMR}{EI}. \tag{17'}$$

If the results of tests are available in which ϵ_1 and ϵ_2 have been determined by measurements, we can also find the stresses directly from (7). We denote the observed strains by ϵ_{01} and ϵ_{02} and the stresses calculated from them, hereafter referred to as the "*observed stresses*," by p_{01} and p_{02} and have:

$$\left.\begin{aligned} p_{01} &= \frac{m^2 E}{m^2-1}\left[\epsilon_{01}+\frac{\epsilon_{02}}{m}\right] \\ p_{02} &= \frac{m^2 E}{m^2-1}\left[\epsilon_{02}+\frac{\epsilon_{01}}{m}\right] \end{aligned}\right\}, \tag{22}$$

It is here to be noted that when the bends are tested in a machine and also generally in practice, the bending moment M is associated with compressive end forces P which produce compressive strains. These strains are included in ϵ_{01} and ϵ_{02} but not in ϵ_1 and ϵ_2 as given in (18) and (19). If then we want to compare the observed and the calculated stresses we must make corrections for the compressive stress p_c produced by P. This is readily done, especially for the section at the top of the bend, where the tangent to the pipe axis is parallel with P. We have here:

$$p_c = -\frac{P}{4\pi rh}.$$

The transverse stress p_{02} will not be affected by p_c.

In practice the pipe is generally under a steam pressure $\bar{p}$, in which case a tensile hoop or transverse stress comes to exist throughout the wall of the pipe and this stress must then be applied with proper sign to p_{02} (that is removed from p_{02}) in order to effect a comparison with p_2. It does not affect the longitudinal stress p_{01}, but in high-pressure pipe lines it may appreciably augment the transverse stress.

According to the most recent theory of strength, the point at which plastic flow begins in an isotropic ductile material in a combined state of stress is given by:

$$p_1^2+p_2^2+p_3^2-p_1p_2-p_2p_3-p_1p_3=p_0^2 \tag{23}$$

which for plane stress becomes:

$$p_1^2 - p_1 p_2 + p_2^2 = p_0^2 \tag{24}$$

where p_0 is the stress at the yield point.

We shall put the expression on the left side of this equation equal to p_e^2 and call p_e the "equivalent stress." Actually p_e does not exist as a stress, being simply a function of p_1 and p_2; but so long as p_e is smaller than p_0 the material is still in the elastic state, when it exceeds p_0 it is in the plastic state. In a later chapter the theory of strength will be discussed and the derivation of these equations explained.

C. Tests Made at the Massachusetts Institute of Technology in 1928

These experiments were made in the Testing Materials Laboratory of the Institute and fall in two groups.

Tests made in the early part of 1928. Two bends of seamless drawn steel pipe were tested, one of 9½-in. and one of 14-in. nominal diameter. Both were placed at the disposal of the Institute for this purpose by the manager of the Boston Navy Yard, Capt. C. M. Simmers (CC), U. S. N.

The large size of the pipes as compared with the 6-in. pipe bend tested in 1926 allowed greater accuracy to be attained in measuring the strains, and by applying a new method for obtaining the transverse stresses as also by carrying the tests to destruction some novel and interesting results were obtained.

The form of the pipe axis and the radius of curvature in the unstrained state were accurately determined by measurements made with steel tape of the distances between a number of punch marks on the pipes.

The bends were set up in the 300,000 lbs. Emery hydraulic testing machine in the Materials Testing Laboratory of the Institute, the pipes being suspended from overhead cranes so that the plane of the bends was vertical. The pressures were transmitted through knife edges permitting rotation of the ends of the pipes in the plane of the bend, and in order to facilitate the fitting and adjustment of the knife edges, the ends of the pipes

were provided with flanges formed of rectangular steel plates welded to the pipes in a plane normal to the axis of symmetry of the bend. Vertical face plates were attached to the heads of the pistons of the testing machine, and between these face plates and the edges of the rectangular flanges the knife edges were inserted. The flat side of the knife edges rested against the machined edge of the flanges, while the sharp edge fitted into a horizontal V-shaped

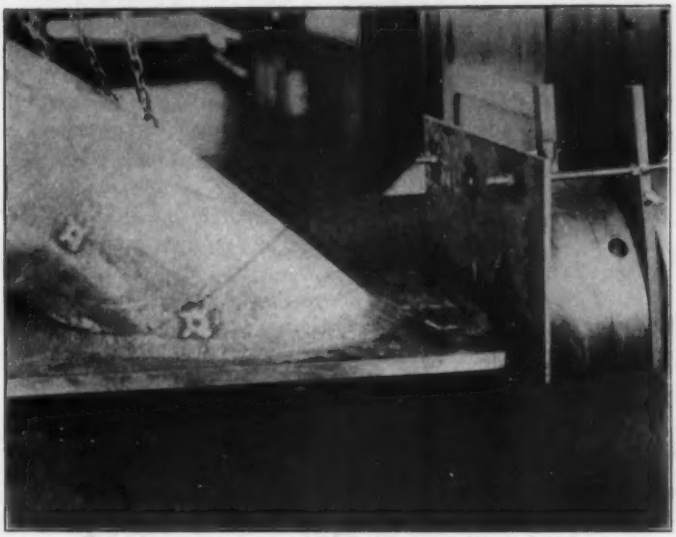


FIG. 3

14-in. pipe set up in the machine

groove, $\frac{1}{8}$-in. deep, cut into the face plate. The rectangular flanges were in the same plane so that the forces P exerted at each end by the pistons were acting along the line of intersection between that plane and the plane of the bend. The arrangement is shown in Fig. 3.

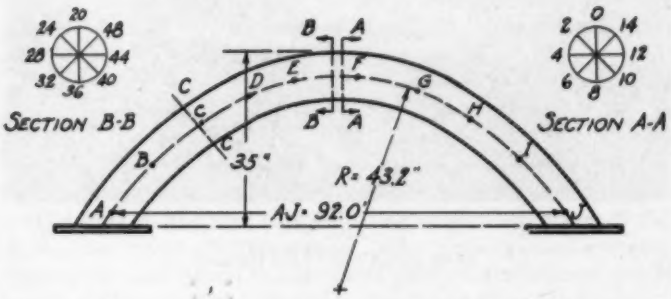
Since the flanges were free to rotate about the knife edges, practically no terminal couples existed and the bends could be regarded as "hinged" at both ends. The bending moment at any

point of the pipe axis was simply Py where y is the ordinate of the point above the line of action of the P-forces, which was taken as the X-axis.

Tests made in October 1928. These tests were made with the 6-in. Walworth pipe bend previously experimented with as described in the 1926 paper. At that time the bend was first subjected to loads up to 2,000 lbs., but as this load produced a permanent set of 1.12 in., the load was not increased beyond 1,500 lbs. in the following tests which were those used in the analysis of 1926. The object of testing this pipe again was to determine more exactly the point at which a permanent set of the bend occurred and also the load at which Lüders' lines began to appear.

FIG. 4

$9\frac{1}{2}$ IN PIPE BEND.
AVERAGE OUTSIDE DIAMETER 9.56"



Description of the Tests

The 9½-in. Pipe Bend. Fig. 4 shows the general shape of the pipe. The length measured along the axis was 110 in., the diameter to the middle surface was 9.305 in. and the average thickness calculated on the basis of weight was: $2h = .255$ in. The radius of curvature at the top of the pipe was $R = 43.2$ in. so that the ratio

between the radius of the bend and the diameter of the pipe was about 4.6.

Two adjacent sections, AA and BB, near the top of the bend, were chosen for strain measurements. Section AA was divided into sixteen equal parts by punch marks on the outside of the pipe. At each mark the scale was knocked off, after which the surface was filed smooth, and with a micrometer caliper the measurements of the outside diameter were obtained between diametrically opposite points. From these measurements the transverse strains were derived as explained below.

On section BB, about 2 in. from AA were marked off 32 equidistant points at each of which two holes were drilled, spaced 2 in. apart in a longitudinal line. These holes served for taking measurements of the longitudinal strains with Berry strain gages.

A third section CC was prepared on about quarter length of the pipe from one end and on this were measured major and minor diameters as well as longitudinal strains at the ends of these diameters.

An initial load of 3,000 lbs. was applied on the pipe in order to eliminate small irregular adjustments at the face plates and knife edges and this condition was used as reference point for all subsequent measurements and for the calculation of strains and deflections.

The load was progressively increased in steps of 1,000 lbs., returning at intervals to the initial condition. Above 7,000 lbs. the load was increased very slowly and cautiously so as to obtain a fairly accurate determination of the limit of proportionality. The changes in the length of the base AJ (Fig. 4), which were measured with a steel tape, remained proportional to the increment of the applied loads up to a load of 10,500 lbs. Above that the linear relation ceased to hold and scale was observed to begin to crack off at about 11,000 lbs.

At 11,500 lbs. well defined Lüders' lines began to appear on the underside of the pipe, inclined at 45 degrees to the axis. At 11,800 lbs. they began to form on the upper side. Pictures of some of the Lüders' lines are shown on Figs. 21 and 22.

The testing machine always tended to maintain a certain load or pressure to which it was set, and up to 12,000 lbs. the pipe was

able to hold the load, that is, the length AJ remained unaltered for a constant load, but at 12,000 lbs. a definite creeping was observed and thereafter the pipe began to yield steadily while the load fell off. The length AJ decreased, the machine always following up, attempting to exert the pressure to which it was set. It was clear that the pipe had commenced to collapse. The

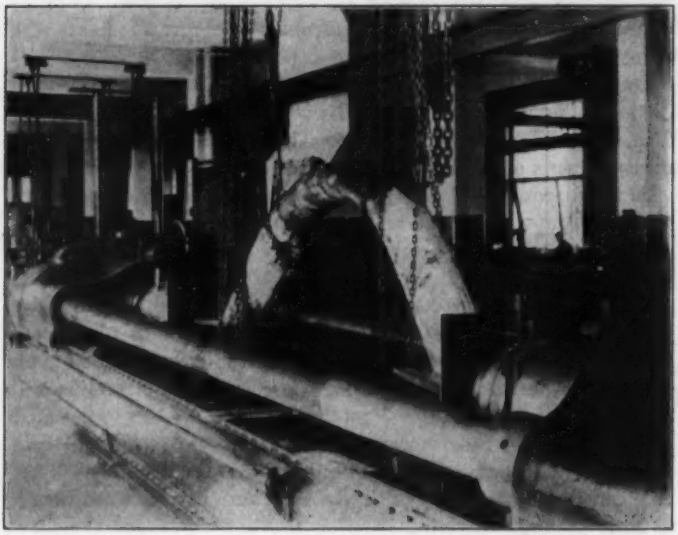


FIG. 5

9½-in. pipe in the machine after the test, knife-edges removed

test was stopped when the load had dropped to 8,000 lbs. A well-defined fold had then appeared about at section AA on the concave side. Fig. 5 shows the bend in the machine after the test when the knife edges had been removed.

Later, the ends of the pipe were forced together in a vertical testing machine as far as was practicable in order to ascertain where actual fracture would occur, but although the pipe was bent back upon itself until the angle between the two parts was

about 30 degrees, as may be seen from Fig. 6, there was no sign of fracture anywhere.

FIG. 6

9½-in. pipe bent back upon itself after the test

The 14-in. Pipe Bend. Fig. 7 shows the general shape of the pipe. At the top of the bend a section XX was marked off, 32 sets of holes were drilled in pairs 2 in. apart for measurement of longitudinal strains as on the 9½-in. pipe, and 12 equidistant spots were prepared for measuring the outside diameters with micrometer caliper. Two inches from section XX, holes were drilled to enable 12 transverse strain measurements to be taken with Berry strain gages around the pipe. A general view of the pipe in the machine is shown in a photograph on Fig. 8 taken after the test when the pressure was removed.

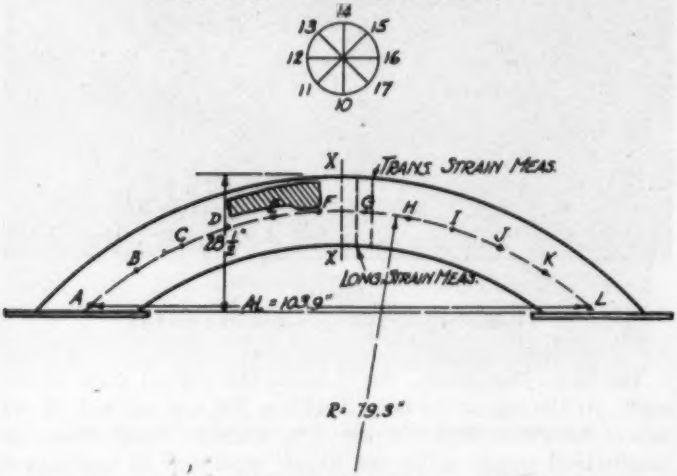
The radius of curvature at section XX was $R=79.3$ in. and the diameter of the pipe reckoned to the middle surface was

13.47 in., so that the ratio between the radius of the bend and the diameter of the pipe was about 5.9. The average thickness of the pipe wall was $2h = .379$ in.

An initial load of 5,000 lbs. was used as a basis or reference point for all subsequent measurements and calculations of strains and deflections.

The load was at once increased to 20,000 lbs. gross and readings

FIG. 7
14 IN. PIPE BEND
AVERAGE OUTSIDE DIAMETER 13.85"
NOTE: HATCHED AREA INDICATES PIECE CUT OUT FOR ETCHING AND EXAMINATION OF LÜDER'S LINES



taken. From this point on the load was increased in steps of 10,000 lbs.

The changes in the base length AL remained approximately proportional to the increments of load up to and including 50,000 lbs. gross. Above that load proportionality ceased. Scale began to crack off just before the load of 50,000 lbs. was reached. At this load a few definite Lüders' lines were observed on the under side of the pipe running at 45 degrees to the principal stresses.

The load was then increased to 60,000 lbs., at which the Lüders' lines became very pronounced, both on the under and on the upper side of the pipe. These lines are seen very clearly on Fig. 23, to the left of section XX. At 60,000 lbs. scale cracked off continuously, the pipe could not hold the load and the length AL decreased steadily. Creeping of the end continued, a fold appeared

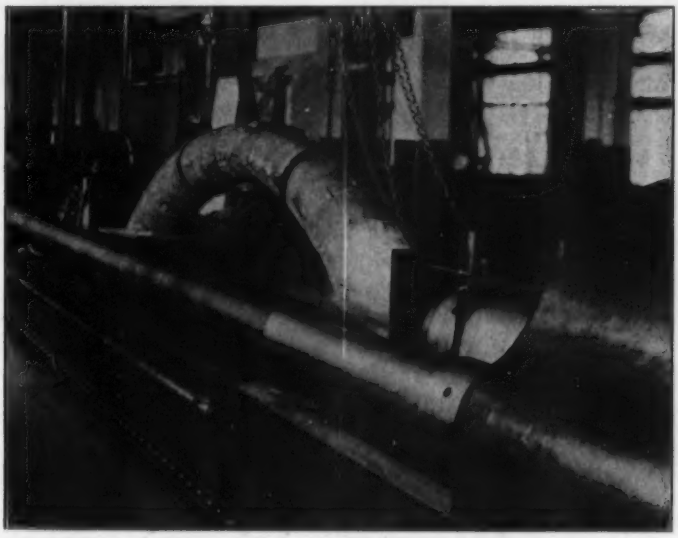


FIG. 8

14-in. pipe in the machine after the test, knife edges removed

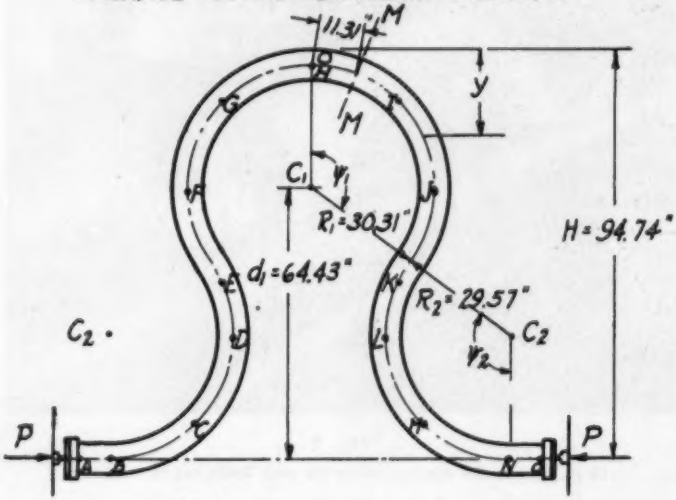
at the middle of the pipe on the under side, the pipe clearly began to collapse and the pressure was taken off.

The 6-in. Walworth Pipe Bend. As stated above, this bend had been tested in 1926, and was then subjected to a load of 2,000 lbs., at which it showed a permanent set. The subsequent tests at that time were carried out under loads not exceeding 1,500 lbs. It was now, in 1928, desired to study more closely the

conditions in the neighborhood of the yield point, but it was found that the load could be increased to about 2,250 lbs. before a permanent set began to appear.

The general form and dimensions of the bend are given in Fig. 9 and are described in more detail in the 1926 paper. It shall here merely be stated that it was lyre-shaped, and provided with caps at the ends. In each cap was a groove placed normal to the plane

FIG. 9
THE 6" WALWORTH PIPE BEND
AVERAGE OUTSIDE DIAMETER 6.656 IN.



of the bend, serving to hold the knife-edges against which pressed the pistons of the testing machine. Observations of longitudinal strains and of the diameters were made at a section MM near the top of the bend, where 32 pairs of holes were drilled 2 in. apart and 4 spots were filed smooth in order to serve for caliper measurements of the diameters in the plane of the bend and normal to the bend.

The radius of curvature at section MM was $R=30.31$ in., and

the average diameter reckoned to the middle surface was 6.39 in. so that the ratio between the radius of the bend and the diameter of the pipe was about 4.74. The average thickness of the pipe wall was .265 in.

An initial load of 300 lbs. was used as in the 1926 experiments. The load was at once increased to 1,500 lbs., which was known to be within the elastic limit, and readings were taken of the longitudinal strains, the two principal diameters at section *MM*, and of *AO*, the distance between the ends of the pipe. Measurements were also made of the distances from *A* and *O* to the top point of the bend *H*. The load was again reduced to 300 lbs. and readings taken. Next the load was increased to 1,500 lbs. and then successively augmented by increments of 100 lbs. at each of which points *AO* was measured. At a load of 1,800 lbs. a complete set of measurements was again taken. The reduction in *AO* was from 3/16 in. to 5/16 in. per 100 lbs. and remained constant until the load reached 2,200 lbs. Here the load was again reduced to 300 lbs., but it was found that there was still no permanent set. Now it was attempted to raise the load to 2,300 lbs., but soon a marked shortening of *AO* took place and the pressure began to fall off, the pipe being unable to hold the load. It is estimated that the bend reached its maximum power of resistance and failed at a load of about 2,250 lbs.

When *AO* had been reduced by about one foot and the pressure in the machine had fallen to 2,000 lbs., Lüders' lines were observed between section *MM* and *I* from the angle $\theta = 0°$ to $\theta = +70°$. At the same time the pipe began to flatten visibly near the *I* section, evidently due to a local weakness. Otherwise the pipe preserved its form and no sign of Lüders' lines was to be detected elsewhere. At this point the pressure was taken off.

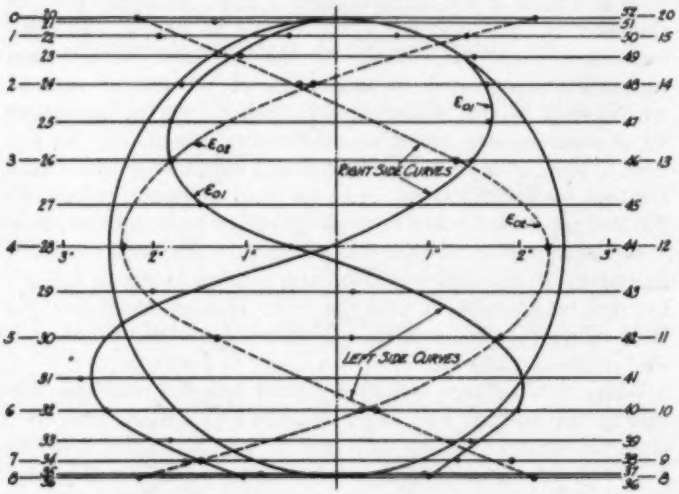
Measurement of the Strains

The *longitudinal strain* measurements were made with Berry strain gages over 2-in. span as explained above, and were reduced to unit strains by simply dividing the elongation or contraction in inches by two. The observed strains ϵ_{01} so obtained include the compressive strains due to end load.

The *transverse strains* ϵ_{02} were not obtained by direct measurement as in case of the longitudinal strains. Direct measurements were indeed taken on the 14-in. pipe around a section about 2 in. from XX, but due to the strong curvature of the sections the calculation of the strains from the gage measurements

FIG. 10

$9\frac{1}{2}$ IN. PIPE BEND
OBSERVED LONGITUDINAL AND TRANSVERSE STRAINS ϵ_{01} AND ϵ_{02}
END LOAD 9000 LB.



OBSERVED LONGITUDINAL STRAIN (FULL) SCALE:- 1" = 0.0003
OBSERVED TRANSVERSE STRAIN (DOTTED) SCALE:- 1" = 0.0004

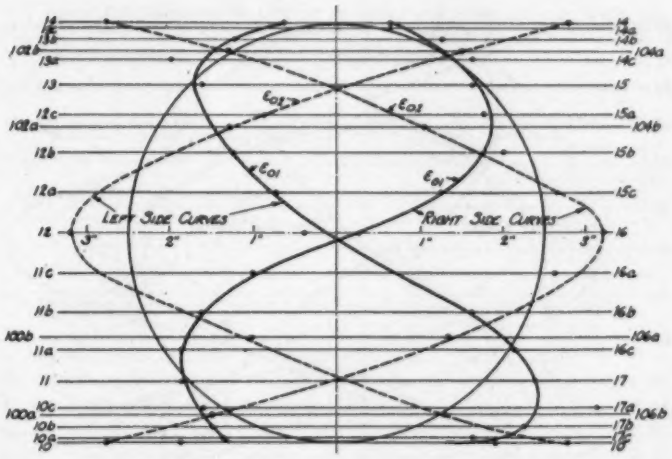
NOTE:- The strains are obtained from the difference between the readings of the strainmeters at loads of 9000 lb. and 3000 lb. The longitudinal strains include the effect of the compression due to the end load of 9000 lb.

was very uncertain and laborious as explained in the analysis of the 1926 tests, whence the results were used as a check only. It was found simpler and more accurate to calculate the transverse strains from measurements of the diameters taken with a micrometer caliper on the sections prepared for this purpose. In

this way a complete picture of the form of those sections was obtained before and after the load was applied. The method gave very consistent results and since it has not, to the author's knowledge, been used before for this particular purpose we shall here describe it in some detail.

FIG. 11

14 IN. PIPE BEND
OBSERVED LONGITUDINAL AND TRANSVERSE STRAINS, ϵ_{01} AND ϵ_{02}
END LOAD 50,000 LB.



OBSERVED LONGITUDINAL STRAIN (FULL) SCALE:- 1" = 0.0004
OBSERVED TRANSVERSE STRAIN (DOTTED) SCALE:- 1" = 0.0004

NOTE:- The strains are obtained from the difference between the readings of the strainmeters at loads of 50,000 lb. and 5000 lb. The longitudinal strains include the effect of the compression due to the end load of 50,000 lb.

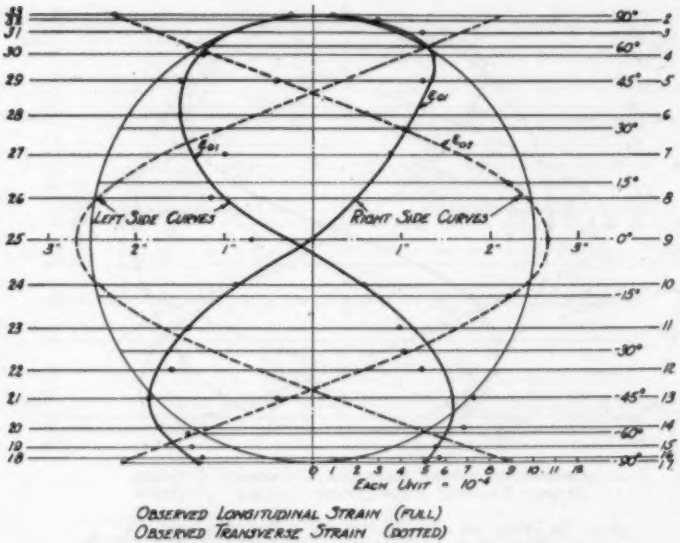
The section at the top of the pipe in the unstrained condition was slightly oval in both bends with major axis horizontal (normal to the plane of bending). It could without sensible error be assumed to be elliptic and on this assumption the radius of curvature was calculated from the formula:

$$r = a^2 b^2 \rho^3 \left[\frac{\cos^2\theta}{a^4} + \frac{\sin^2\theta}{b^4} \right]^{\frac{3}{2}} \qquad (25)$$

where ρ and θ are polar coördinates with the center of the ellipse as origin; a and b are the major and minor axes respectively. The axes as well as the radius vector ρ were obtained from the caliper measurements. The axis of reference was so placed that

FIG. 12

6-IN. PIPE BEND
OBSERVED LONGITUDINAL AND TRANSVERSE STRAINS ϵ_{01} AND ϵ_{02}
END LOAD 1800 LB.



OBSERVED LONGITUDINAL STRAIN (FULL)
OBSERVED TRANSVERSE STRAIN (DOTTED)

NOTE:- The strains are obtained from the difference between the readings of the strainmeters at loads of 1800 lb. and 300 lb. The longitudinal strains include the effect of the compression due to the end load of 1800 lb.

when $\theta=0$, $\rho=a$ and when $\theta=\frac{\pi}{2}$, $\rho=b$. Referring to Fig. 2, A is the end of the major axis and B of the minor axis.

When the pipe bend was compressed as in the tests, the ellipse flattened somewhat more, the axes as well as the radii vectores

changed and were measured with the caliper. Applying again formula (25) the new radius of curvature r' could be calculated for every point.

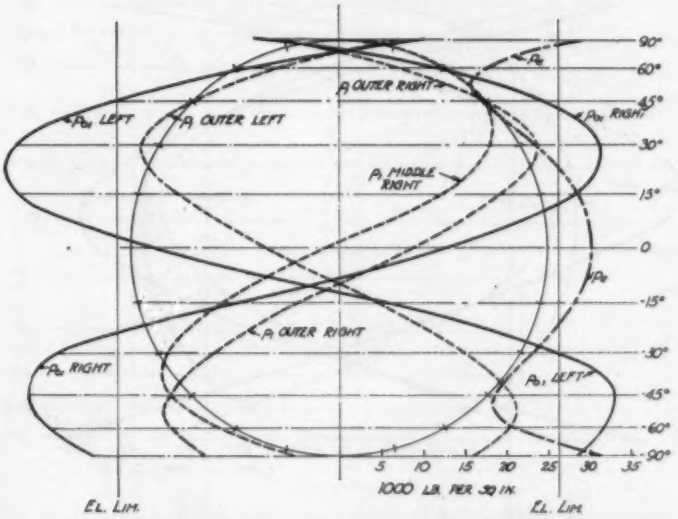
The observed transverse strain could now be found from (9):

$$\epsilon_{02} = z\left(\frac{1}{r'} - \frac{1}{r}\right). \tag{9'}$$

FIG. 13

9½-IN. PIPE BEND

CALCULATED LONGITUDINAL STRESSES AT MIDDLE AND OUTER SURFACE, p_1 (DOTTED)
OBSERVED LONGITUDINAL STRESSES AT OUTER SURFACE, p_{01} (FULL)
EQUIVALENT STRESSES, p_0
END LOAD 9000 LB.

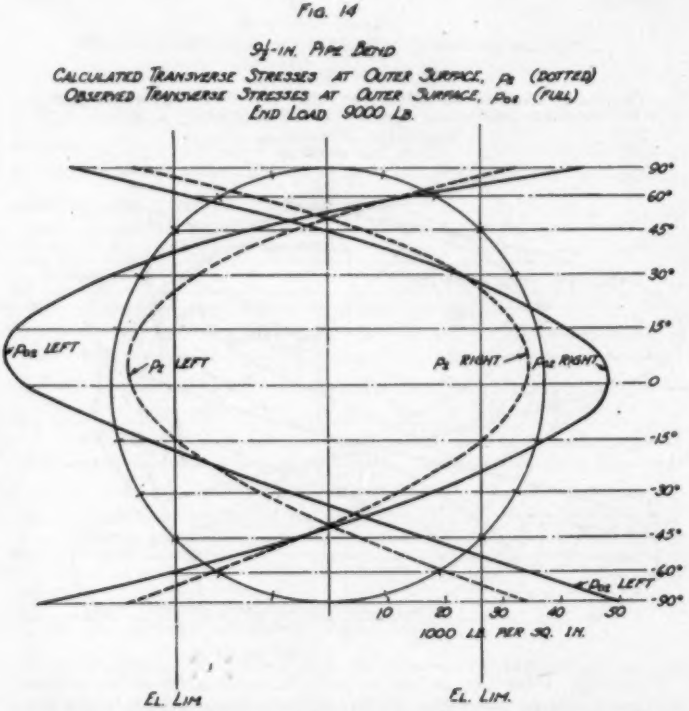


The limits of accuracy of the caliper measurements were estimated at ±.001 in., which on a diameter of 9½ in. or 14 in. gives a much higher accuracy than obtainable with direct strain measurements.

In order to obtain a complete picture of the strain distribution, the observed longitudinal and transverse strains were plotted in curves. Fig. 10 shows the curves for the 9½-in. pipe. The hori-

zontal ordinates give the strains at the proper height above the neutral axis, they represent the differences between the readings at a load of 9,000 lbs. and those at the initial load of 3,000 lbs., but the longitudinal strains include the compressive strain due to the end load of 9,000 lbs. Ordinates to the right indicate tensile

FIG. 14

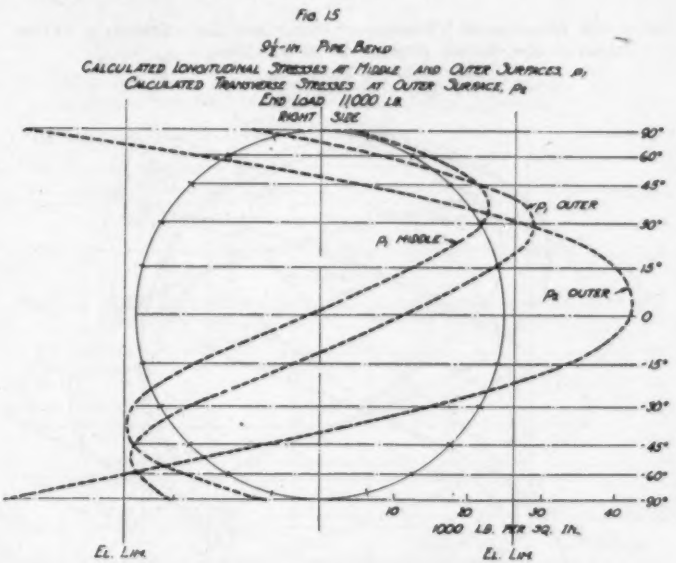


strains for the right side of the section and compressive strains for the left side and *vice versa*. The observed spots are marked off and fair curves are drawn between them, disregarding points which are obviously in error.

Similar curves are shown in Fig. 11 for the 14-in. pipe under a load of 50,000 lbs., but as the initial load was here 5,000 lbs.,

the strains actually correspond to a load of 45,000 lbs. including, however, the compressive stress due to an end load of 50,000 lbs.

The strains in the 6-in. pipe are shown in Fig. 12. The load was here 1,800 lbs. and the initial load 300 lbs., so that the strains correspond to a load of 1,500 lbs. The compressive stress due to the end load is insignificant in this case.



D. Analysis of the Tests

The Stresses

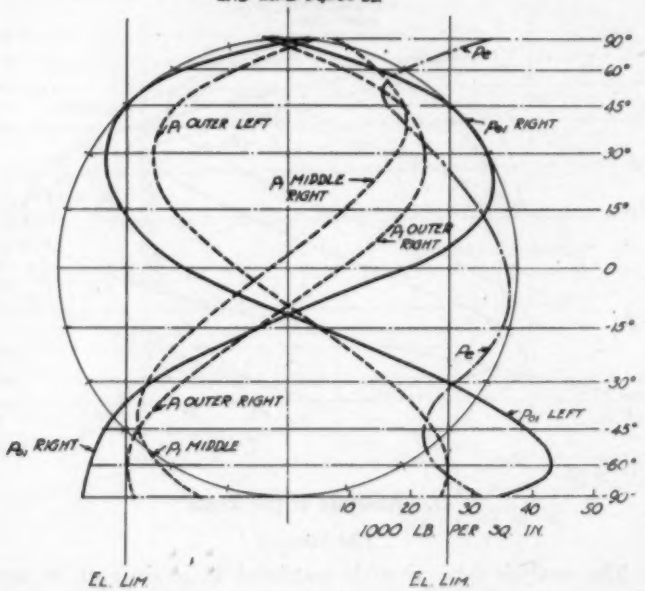
The analysis here given is restricted to sections at or near the top of the bends, where the bending moments were greatest and where an overstrain was likely to occur first. In the 9½-in. pipe the sections specially considered were *AA* and *BB*, Fig. 4; in the 14-in. pipe *XX*, Fig. 7; and in the 6-in. pipe *MM*, Fig. 9. The measurements taken at section *CC* on the 9½-in. pipe were used merely as a check on those at the top, with which measurements they were found to be quite consistent.

The stresses were calculated by two methods, first from the purely theoretical formulas (20) and (21) and second from the observed strains, applying formulas (22). By plotting the "calculated" and "observed" stresses so obtained in the same dia-

FIG. 16

14-IN. PIPE BEND

CALCULATED LONGITUDINAL STRESSES AT MIDDLE AND OUTER SURFACES p_i (DOTTED)
OBSERVED LONGITUDINAL STRESSES AT OUTER SURFACE, p_{oi} (FULL)
EQUIVALENT STRESSES p_e
END LOAD 50000 LB



gram for each case, it was easy by a comparison to study the accuracy of the theoretical formulas.

Figs. 13, 14 and 15 show such diagrams for the 9½-in. pipe, Fig. 16, 17 and 18 for the 14-in. pipe and Fig. 19 and 20 for the 6-in. pipe.

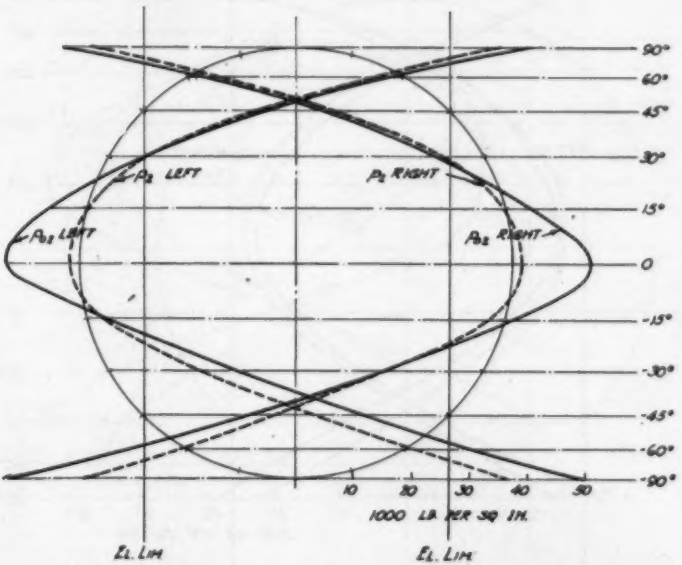
Poisson's ratio m was reckoned equal to 10/3. The modulus

of elasticity E was determined by coupon tests on pieces cut from scraps left from the 14-in. pipe after it was cut off for fitting the rectangular flange plates. It was found that $E=32{,}000{,}000$ lbs. per sq. in. The proportional limit of the material in simple tension was found at 26,300 lbs. per sq. in.

FIG. 17

14-IN. PIPE BEND

CALCULATED TRANSVERSE STRESSES AT OUTER SURFACE, p_2 (DOTTED)
OBSERVED TRANSVERSE STRESSES AT OUTER SURFACE, p_{02} (FULL)
END LOAD 30,000 LB.



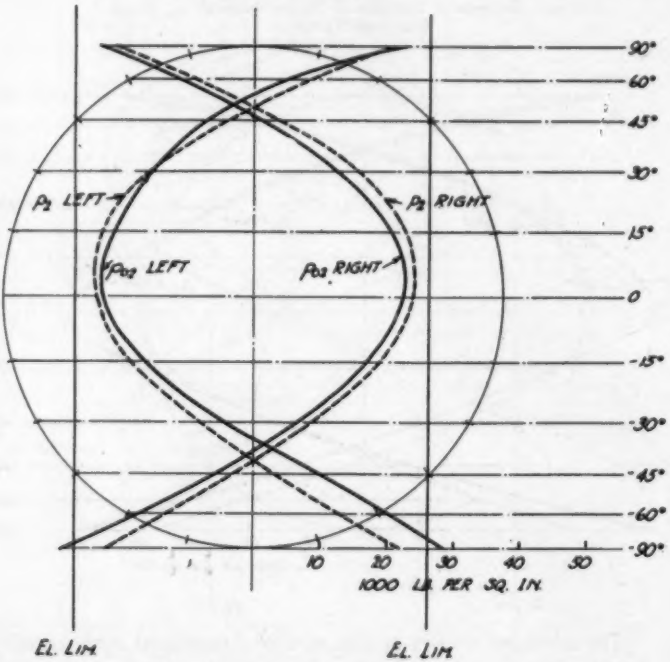
The calculated stresses p_1 and p_2 were determined corresponding to actual or gross load P. In case of the longitudinal stresses it was necessary to correct for the compressive stress p_c due to end load, which is not included in formula (20), but which enters into the observed stresses with which the calculated stresses are to be compared. Consider, for instance, the 9½-in. pipe under a load of

9,000 lbs. Here the bending moment is $M = 9{,}000 \times 30.2 = 271{,}800$ in. lbs., which is used in (20), but to the value of p_1 so obtained must be applied the direct compressive stress $p_c = -1{,}200$ lbs. per

Fig. 18.

14-in. Pipe Bend

Calculated Transverse Stresses at Outer Surface, p_1 (dotted)
Observed Transverse Stresses at Outer Surface, p_{01} (full)
End Load 30,000 lb.



sq. in. produced by the end load $P = 9{,}000$ lbs. The resulting stress is plotted on Fig. 13 in the curves denoted: "p_1 outer."

There is also plotted a curve for the calculated longitudinal stress at the middle surface, obtained from (20) by omitting the

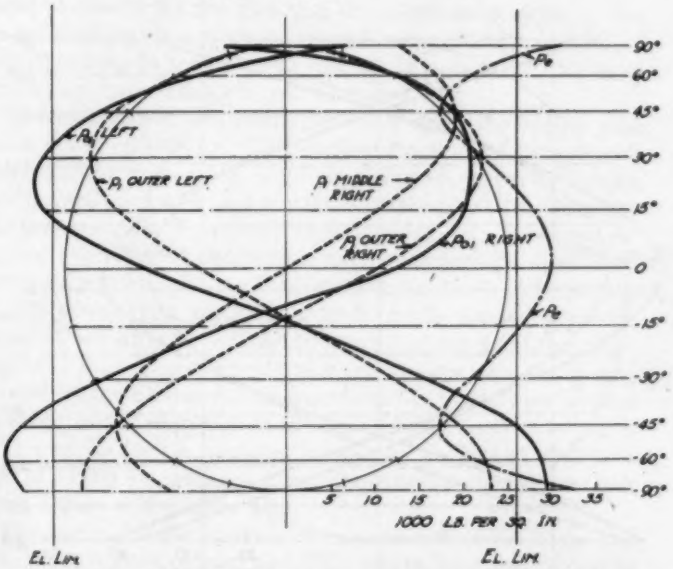
last term inside brackets and then applying p_c. This curve is denoted "p_1 middle."

The calculated transverse stresses at the outer surface for the 9½-in. pipe are obtained from (21) with $P = 9{,}000$ lbs. and are plotted in Fig. 14, but no correction is necessary in this case, as

FIG. 19

6-IN. PIPE BEND

CALCULATED LONGITUDINAL STRESSES AT MIDDLE AND OUTER SURFACES p_1 (DOTTED)
OBSERVED LONGITUDINAL STRESSES AT OUTER SURFACE, p_{01} (FULL)
EQUIVALENT STRESSES p_0
END LOAD 1800 LB.



the transverse stresses are independent of p_c. They are denoted by p_2.

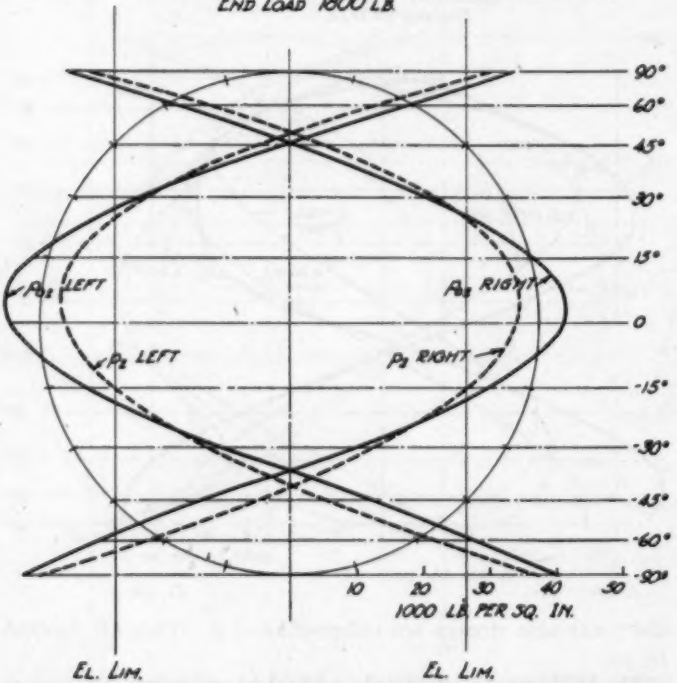
Fig. 15 shows the calculated longitudinal and transverse stresses for the 9½-in. pipe under an end load of 11,000 lbs., at which the elastic limit had been passed and a small permanent set of the bend was observed.

For the 14-in. pipe the calculated longitudinal and transverse stresses were obtained in the same way and are shown in Figs. 16 and 17 respectively for a load of 50,000 lbs. In Fig. 18 are shown the calculated transverse stresses for the same pipe at a load of 30,000 lbs.

FIG. 20

6-IN. PIPE BEND

CALCULATED TRANSVERSE STRESSES AT OUTER SURFACE, p_2 (DOTTED)
OBSERVED TRANSVERSE STRESSES AT OUTER SURFACE, p_{02} (FULL)
END LOAD 1800 LB.



Finally the calculated stresses for the 6-in. pipe are shown in Figs. 19 and 20, but in this case it was unnecessary to make any correction for stresses due to end pressure, because the load was only 1,800 lbs.

The *observed stresses*, p_{01} and p_{02}, as obtained by means of formulas (22) from the observed strains, were not actual or total stresses, since the strains were only relative to a certain initial condition. Now, it was desired in particular to study the state of stress and the behavior of the bends in the condition where flow of the material began to take place, and it was therefore necessary to determine the stresses which actually existed and to compare them with the calculated stresses and the deflections of the pipe. Hence, the observed stresses as obtained from formula (22) were augmented to correspond to the actual or gross load, but in so doing it was necessary in case of the longitudinal stresses to make allowance for the fact that the compressive stress p_c due to end load was not to be augmented, being already included to its full amount in the observed strains. In case of the transverse stresses no correction for p_c was necessary.

In illustration of this method consider again the 9½-in. pipe under a load of 9,000 lbs. The observed longitudinal stresses were first obtained by means of (22) from the strain curves on Fig. 10, which correspond to a load of 9,000−3,000=6,000 lbs. Then the compressive stress due to end load, that is, $p_c=-1{,}200$ lbs. per sq. in. was removed, whereupon the remainder was augmented by 50 per cent to obtain the stress at 9,000 lbs. due to pure bending. Finally p_c was again applied in order to get the actual "observed" stress. The figures so obtained are plotted in Fig. 13 in the curves denoted "p_{01}." The observed transverse stresses were simply augmented for the increase in load from 6,000 lbs. to 9,000 lbs. and the resulting curves were plotted in Fig. 14, being denoted by p_{02}.

The curves for the observed stresses for the 9½-in. pipe at a load of 11,000 lbs. were not plotted because the strains were here far in excess of the elastic limit.

The curves for the observed stresses in case of the 14-in. and 6-in. pipes were calculated as for the 9½-in. pipe and plotted in Figs. 16 to 20.

Discussion of the Stress Curves. The curves for the observed longitudinal stresses show in all cases the same characteristics as those for the calculated stresses, conforming to the theory that takes into account the flattening of the transverse sections. Only

a very small tensile stress or even a compressive stress is found at the top of the section, where, according to the old theory of bending, the tensile stress should be a maximum. Actually the observed maximum tensile stresses on the outer surface are found at 25 to 30 degrees from the neutral axis, at the point where they should be according to the new theory when the influence of the transverse strains and the compression due to end load are taken into account. The compressive stresses at the bottom of the section, observed as well as calculated, are greater than the tensile stresses at the top, yet, generally, they have their maximum at some -45 to -60 degrees from the neutral axis.

The curves for the observed *transverse stresses* show the same agreement with the characteristics of those for the calculated transverse stresses as in case of the longitudinal stresses.

The only marked discrepancy between the curves for observed and theoretical stresses is found at loads near the limit of proportionality of the pipes, where the observed maximum stresses, longitudinal as well as transverse, are considerably greater than the calculated stresses. This discrepancy, which did not occur at more moderate loads (see Fig. 18), cannot be accounted for by errors in the observations, for on the whole the observed results were quite consistent, but before this matter is further discussed, attention is invited to Tables 1 and 2, in which a summary is given of the maximum stresses as obtained from the curves.

An inspection of Table 1 shows that in the 9½-in. pipe at a load of 9,000 lbs. and the 6-in. pipe at 1,800 lbs., where we know that the deflection of both pipes was well within the limit of proportionality, the calculated maximum *longitudinal stresses* were also much below 26,300 lbs. per sq. in., the elastic limit of the material. Yet the observed maximum longitudinal stresses were far in excess of that limit. In the 14-in. pipe at a load of 50,000 lbs. the elastic limit was closely approached by the calculated longitudinal stress on the side of compression and here the observed stress in compression was 43,000 lbs. per sq. in.

For the 9½-in. pipe at a load of 11,000 lbs. the calculated longitudinal stress exceeded the elastic limit, although not by more than about 2,000 lbs. per sq. in., but the observed stress,

TABLE 1

Maximum Longitudinal Stresses p_1 and p_{o1} in Pounds per Square Inch

	9½-in. Bend		*14-in. Bend*	*6-in. Bend*		
	9,000	11,000	50,000			2,250
Load in Lbs.	*Failure at* 10,500		(*Failure*)	1,800	2,000	(*Failure*)
Tension, Calculated Stress Middle Surface	18,500	22,600	19,400	19,300	21,400	24,100
Tension, Calculated Stress Outer Surface	23,800	28,700	22,200	22,300	24,800	27,900
Tension, Observed Stress Outer Surface	41,000	Excessive	32,600	28,500		
Compression, Calculated Stress, Middle Surface	21,100	25,800	24,800	19,300	21,400	24,100
Compression, Calculated Stress, Outer Surface	21,000	25,700	26,100	23,000	25,600	28,800
Compression, Observed Stress, Outer Surface	37,200	Excessive	43,000	31,900		

TABLE 2

Maximum Transverse Stresses p_2 and p_{o2} in Pounds per Square Inch

	9½-in. Bend		*14-in. Bend*		*6-in. Bend*		
Load in Lbs.	9,000	11,000	30,000	50,000	1,800	2,000	2,250
Tension, Calculated Stress near Neutral Axis	34,500	43,000	24,000	39,000	34,500	38,300	43,100
Tension, Observed Stress near Neutral Axis	54,800	Excessive	22,700	51,000	42,900		
Compression, Calculated Stress at Lowest Point	−36,300	−44,300	−22,100	−41,100	37,600	41,800	47,000
Compression, Observed Stress at Lowest Point	−50,300	Excessive	−28,900	−50,000	40,200		

determined on the basis of the observed strains, was of the order of 85,000 lbs. per sq. in., both in tension and in compression.

From Table 2 it is seen that the calculated *transverse* stresses reach very high values long before the pipe as a whole shows a permanent set. In fact, in the 9½-in. pipe the calculated transverse stress reaches the elastic limit at a load of about 7,000 lbs.; in the 14-in. pipe at a load of about 32,000 lbs.; and in the 6-in. pipe at a load of 1,300 lbs. The observed stresses at loads where the pipe is near the elastic limit are of the order of 50,000 lbs. per sq. in.

That such high stresses did not actually exist is evident from the fact that the stress couple calculated from them is far in excess of the actual bending moment. The excessive stresses simply indicate that the limit of plasticity had been passed. In the plastic state a ductile metal like iron or steel behaves essentially as a viscous almost incompressible fluid, and at first remains nearly at the elastic limit. When it is drawn out it hardens and the elastic limit is raised, but in the present case, where plasticity at first existed in limited zones, surrounded by elastic metal, it appears that there could not be much drawing out of the metal, although the strain was sufficient to indicate much higher stresses than actually existed.

As stated in Chapter B, the limit of plasticity is reached when a certain function of the principal stresses p_e, here called the *equivalent stress*, attains the value of p_0, the elastic limit or the yield point of the material in simple tension.[8] In Figs. 13, 16 and 19 are shown curves representing the equivalent stress at the outer surface, calculated from formula (24). When these curves fall beyond the straight vertical lines, which on the diagrams represent the elastic limit, $p_0 = 26{,}300$ lbs. per sq. in., the material should be in the plastic state and in that case the stress would probably not much exceed that limit. It is seen that in all three pipes the equivalent stress exceeds the elastic limit near the neutral axis as well as at the top and bottom of the section under conditions where the calculated longitudinal stress is still moderate and the pipe as a whole does not yet show a permanent set.

[8] It is not here attempted to distinguish between the elastic limit and the yield point.

The maxima of the equivalent stress are found in the same regions as the maxima of the transverse stress. The curves for the equivalent stress at the inner surface are similar to those for the outer surface, but the maxima on the sides occur at about $\theta = +15°$ and are more pronounced.

The discrepancy between the calculated and observed stresses may be thus explained, at least partly. The plastic state was evidently reached first on the inner and outer surfaces over limited areas. Gradually as the load increased, these areas became larger and a redistribution of the stresses took place whereby greater strains would occur in adjacent parts. While the plastic zones spread over greater areas of the surface, they would probably also penetrate deeper into the metal towards the middle surface until finally this process would have a measurable permanent effect on the deflection of the bend. In the meantime the measured strains would indicate much higher stresses than actually existed at certain points, while the calculated longitudinal stresses were still within the elastic limit.

Another cause of the discrepancy may be found in irregularities in the pipe. In the bending process the material is exposed to mechanical and heat treatment, which in spite of all care may well produce variations in the wall thickness, especially near the top of the bend.

Deflection and Deformation of the Pipes

Horizontal Deflection of the Ends. The approach of the two ends of the pipe to each other, as determined by direct measurement with steel tape between the end punch marks, was compared with the theoretical value calculated from formula (15) of the first 1928 paper for a pipe bend hinged at both ends:

$$\Delta x = -\frac{P}{EI}\int Ky^2 ds \tag{26}$$

where P is the net load, *i.e.*, the actual load minus the initial load.

The integration extends along the axis from A to J in the 9½-in. pipe, from A to L in the 14-in. pipe, and from A to O in the 6-in. pipe.

TABLE 3

Horizontal Deflection of the Ends of the Pipe, Δx

9½-in. Pipe				*14-in. Pipe*				*6-in. Pipe*			
Gross Load Lbs.	*Net Load Lbs.*	Δx *Measured, Inches*	Δx *Calculated, Inches*	*Gross Load Lbs.*	*Net Load Lbs.*	Δx *Measured, Inches*	Δx *Calculated, Inches*	*Gross Load Lbs.*	*Net Load Lbs.*	Δx *Measured, Inches*	Δx *Calculated, Inches*
3,000	0	0	0	5,000	0	0	0	300	0	0	0
4,000	1,000	.07	.05	20,000	15,000	.10	.09	1,500	1,200	3.19	2.73
5,000	2,000	.13	.11	30,000	25,000	.13	.15	1,800	1,500	3.87	3.41
6,000	3,000	.19	.16	40,000	35,000	.20	.20	2,000	1,700	4.44	3.87
7,000	4,000	.25	.22	50,000	45,000	.28	.26	2,200	1,900	5.06	4.32
9,000	6,000	.40	.32								
11,000	8,000	.63	.43								

Tables 3 gives measured and calculated values of Δx for the three pipes corresponding to the net load.

Especially for the 9½-in. and 6-in. pipes the actual deflections were greater than the calculated already before the pipe as a whole showed a permanent set, a fact which is probably connected with the discrepancy in observed and calculated stresses discussed above.

Deformation of the Transverse Section. The flattening of the transverse section near the top of the bend was determined directly by caliper measurements of the diameters, and one-half of the observed reduction in vertical diameter was compared with the theoretical vertical deflection Δy_B of the top point of the section (see Fig. 2). The formula used in the calculation of Δy_B is developed in the 1926 paper and is given as (17′) page 82 of that paper.

$$\Delta y_B = \frac{KMR}{EI} \frac{120r^5}{(480h^2R^2+101r^4)}. \qquad (27)$$

The results are given in the following table:

TABLE 4

Vertical Deflection of the Top Point B of the Section, Δy_B

Dia. of Pipe	*6-in.*			*9½-in.*		*14-in.*
P, Load Lbs.....	1,500 (1926)	1,500	1,800	9,000	11,000	50,000
Δy_B, Calc., inches.	.024	.025	.030	.061	.075	.095
Δy_B, Meas., inches.	.024	.025	.031	.074	.129	.110

For the 6-in. pipe are entered the deflections for $P = 1{,}500$ lbs. from the 1926 tests in addition to those from 1928. So long as the load is within the point where the bend begins to show a permanent set, the measured value of Δy_B is the same or only slightly greater than the calculated value, but as soon as that limit is passed, as in the 9½-in. pipe at 11,000 lbs., there is a very marked increase in the deformation of the section. This, as explained below, is due directly to plastic flow at the ends of the principal axes of the deformed section.

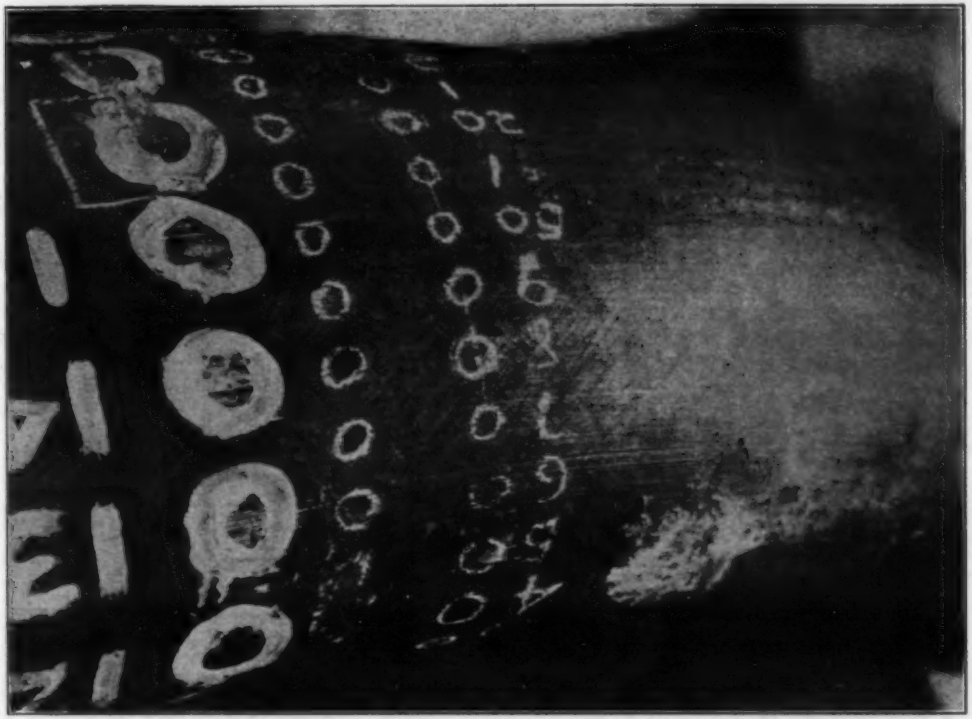


FIG. 21. 9½-in. pipe — Lüders' Lines extend from about $\theta = +29°$ to $\theta = +62°$

Note: The marks for caliper measurements on section *A A* are seen to the left The double line of punch marks are for strain gage measurements, section *BB*. The top of the pipe is at mark 20.

FIG. 22. 9½-in. pipe — Lüders' Lines extend from about $\theta = -22°$ to $\theta = -45°$ and are visible also on upper part of the pipe

Note: The letter F is seen also in Fig. 21, its middle is at about $\theta = 20°$, but the mark which it denotes is at $\theta = 0°$, somewhere in the white blotch.

FIG. 23. Upper part of 14-in. pipe — Lüders' Lines extend from about $\theta = +45°$ to $\theta = +67°$

Note: The marks to the right are those of Section XX. The mark F at the bottom is at the level of the axis $\theta = 0°$. The small mark at the extreme upper right hand corner is at the top of the pipe, $\theta = +90°$.

FIG. 24. Lower part of 14-in. pipe — Lüders' Lines extend from $\theta = -5°$ to $\theta = -45°$
Note: The mark F is at the level of the axis, $\theta = 0°$.

Lüders' Lines

The formation of Lüders' lines, also called strain lines, on the surface of the pipes has been already referred to and photographic pictures of them are given in Figs. 21, 22, 23, 24 and 27.

Fig. 21 shows a pattern of lines near sections AA and BB at the top of the 9½-in. pipe bend. They extend in the upper part of the pipe circumferentially from about $\theta = +29°$ to $\theta = +62°$. Fig. 22 shows lines in the lower part of the 9½-in. pipe in the same region, extending from $\theta = -22°$ to $\theta = -45°$. These lines became visible by the cracking of the mill scale which covered the underside of the pipe. The letter F (Fig. 4), which is seen in both photographs, is marking a spot at $\theta = 0°$ on what is here referred to as the front side of the pipe.

Fig. 23 shows the lines near section XX at the top of the 14-in. pipe bend in the upper part of the pipe. The lines extend from $\theta = +45°$ to $\theta = +67°$. Fig. 24 shows the lines in the lower part of the 14-in. pipe in the same region extending from $\theta = -5°$ to $\theta = -45°$, made visible as in the 9½-in. pipe by cracking of the mill scale. Point F, which is at $\theta = 0°$ on the level of the axis, is visible in both pictures and the marks on section XX are seen in Fig. 23 to the right.

The lines here referred to all run at about 45° to the axis of the pipe, but in the lower part of the pipe on Figs. 22 and 24, below the level of $\theta = -45°$ is seen a confused network of lines, which extends under the bottom of the pipe, in some places running transversely, in others longitudinally. This phenomenon, however, is local and exists only in places where the pipe wall takes a bulge upwards towards the neutral plane.

The patterns of lines here exhibited near the top of the bend on the front side of both pipes is found in the same sectors on the rear side and is repeated at intervals elsewhere except near the ends.

Since the lines run at 45° to the axis and therefore at 45° to the principal stresses, they are in the plane of the principal shear stresses which at any point act in two planes at right angles to each other passing through the radius of the section. According to the maximum shear theory, failure should occur in these planes if the principal stresses p_1 and p_2 are of opposite sign as they are

in certain sectors, and the maximum shear stress on the surface which we denote by q_3 would then be given by:

$$q_3 = \pm \frac{p_1 - p_2}{2} \tag{28}$$

where p_1 and p_2 are the principal stresses on the outer surface as given by (20) and (21). In that case we might expect that the strain lines would appear where q_3 is a maximum.

But if p_1 and p_2 are of the same sign, failure should occur in planes passing through the direction of the smaller of the two stresses, for according to the maximum shear theory the planes of failure must always pass through the axis of the middle stress, of which failure is assumed to be independent. In the present case the radial stress p_3 is equal to zero and thus the smaller of the stresses p_1 and p_2, if they are of the same sign, becomes the middle stress. If, for instance $p_2 > p_1$, as occurs at the bottom of the section where there is a strong compressive transverse stress and a relatively small compressive longitudinal stress, the latter is the middle stress. Hence the planes of failure should pass through the longitudinal tangent to the surface and be inclined at 45° to the tangent plane, and the maximum shear stress would be:

$$q_1 = \pm \frac{p_2}{2}.$$

Thus the strain lines might be expected to run longitudinally in that case as they actually did in certain places. Perhaps failure of this kind may explain the confused network of lines which appeared on the underside of the pipe where buckling took place and where the principal stresses may have acted in various directions although of the same sign.

As explained above, the most regular patterns of lines ran at 45° to the axis, indicating maximum shear stresses q_3 of magnitude as determined by (28) and causing distortion in the tangential plane. It is of interest to examine whether these stresses correlate with the strain lines and for this purpose q_3 was calculated for the 9½-in. and for the 14-in. pipe, the values of p_1 and p_2 being taken from Figs. 13, 14, 16 and 17. The shear stresses so

obtained are plotted in Figs. 25 and 26, where also the location of the strain lines are indicated (by hatching).

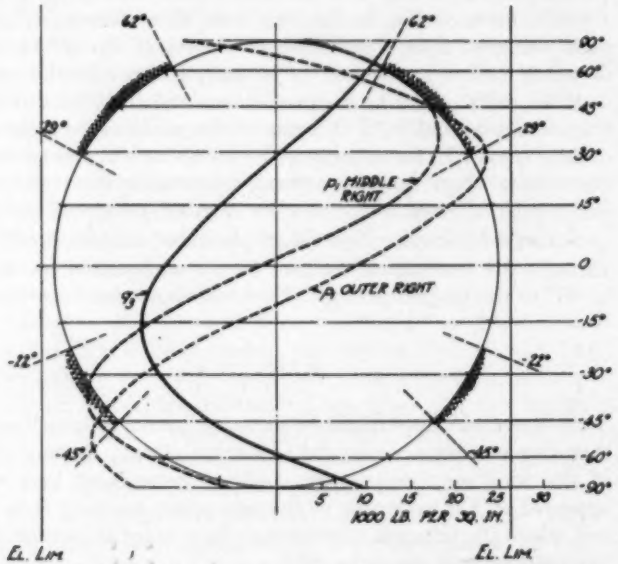
It is seen from Fig. 25 that no correlation whatever exists between the lines and q_3 for the 9½-in. pipe, and for the 14½-in.

FIG. 25

9½-IN. PIPE BEND AT SECTION F

LOCATION OF LÜDER'S LINES

CURVES FOR CALCULATED LONGITUDINAL STRESSES AT MIDDLE AND OUTER SURFACES p_1 CURVE FOR MAXIMUM SHEAR STRESS AT OUTER SURFACE, q_3



pipe, Fig. 26, only at $\theta = -15°$ where q_3 has a maximum, 18,600 lbs. per sq. in., while the lines extend from $\theta = -5°$ to $\theta = -45°$. The shear stress has an extreme at the top of the section but here the lines are quite absent. On the whole, then, there seems to be no satisfactory correlation between the shear stresses and the Lüders' lines.

The two other principal shear stresses q_1 and q_2 are equal to

$p_2/2$ and $p_1/2$ respectively. The former reaches calculated values of nearly 20,000 lbs. per sq. in. tending to produce longitudinal strain lines, the latter tends to produce transverse strain lines but reaches only moderate values. None of these stresses need to be considered in connection with the diagonally placed strain lines and, therefore, are not plotted on Figs. 25 and 26.

An attempt to correlate the strain lines with the equivalent stresses did not meet with better success, for as seen from Figs. 13 and 16 the equivalent stresses have a maximum a little below the neutral axis and extremes at top and bottom, and hence show no correspondence whatever with the strain lines.

So far we have dealt with stresses on the surface and have found no correlation between them and the strain lines. Consider now the middle surface where only a longitudinal stress exists since the transverse stress is zero, and where the maximum shear stress q_3 is equal to $p_1/2$. Since the curve for the shear stress is the same as that for the longitudinal stress in this case, although to a different scale, we need only consider the longitudinal stress, and if the strain lines are determined by the shear stress in the middle surface, we should find a correlation between the lines and the curves for longitudinal stress in that surface. In order to examine this point the curve for the calculated longitudinal stress at the middle surface is reproduced in Figs. 25 and 26, and for comparison also the curve for longitudinal stress at the outer surface is given.

It is seen from Fig. 25 that for the 9½-in. pipe the correspondence between the strain lines and the maxima of the longitudinal stress at the middle surface is almost perfect, although the lines are situated a little higher than the calculated maximum values of the stresses. In case of the 14½-in. pipe this discrepancy is a little more emphasized, the lines being somewhat higher relative to the points of maximum stress. It is to be noted, however, that the lines did not develop until after the pipe had commenced to yield, and that the photographic pictures of the lines were taken after complete breakdown had taken place. At that time notably the 14-in. pipe was excessively flattened along the middle portion of the bend, as seen in Fig. 8, and the stress distribution was not at all as assumed in the calculation for the

stress curves. In fact, we may conclude that, due to the strong bulge inwards (upwards) of the underside of the pipe, allowing this part to shirk its duty in resisting the bending moment, the neutral axis must have moved upwards and actually the longitudinal stresses may well have had their maximum values in

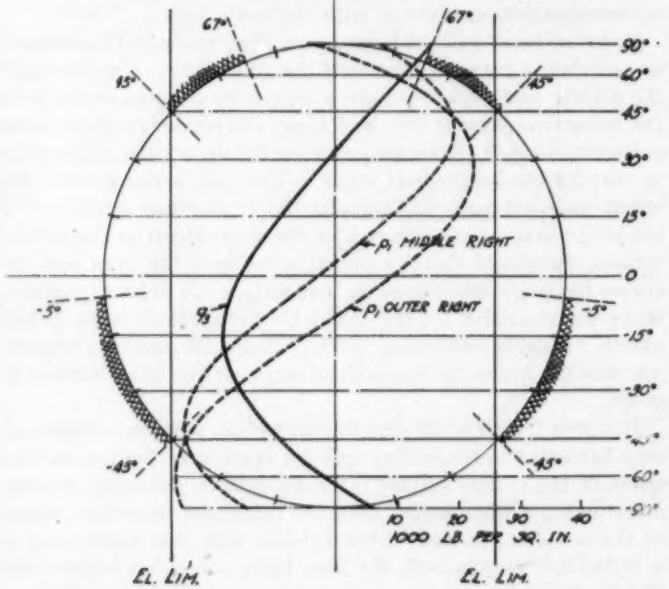
FIG. 26

14-IN PIPE BEND AT SECTION F

LOCATION OF LÜDERS LINES

CURVES FOR CALCULATED LONGITUDINAL STRESSES AT MIDDLE AND OUTER SURFACES, p_1

CURVE FOR MAXIMUM SHEAR STRESS AT OUTER SURFACE, q_3



agreement with the Lüders' lines. If the Lüders' lines depended essentially on the longitudinal stresses in the middle surface, and not merely on surface stresses, we should expect to find that the lines existed not only at the surface but throughout the whole thickness, including the middle surface. It was decided to investigate this point.

For this purpose a piece of approximately 24 in. by 6 in. was cut with an oxy-acetylene torch from the 14-in. pipe wall in the region of the Lüders' lines shown in Fig. 23 and as indicated in Fig. 7. The specimen so obtained was planed down over an area of about 12 in. by 3½ in. until at the thinnest point the thickness of the metal was approximately 40 per cent of the original thickness. In order to facilitate etching, two specimens were cut from the original piece, each of them approximately 6 in. by 3 in. To remove the burnished surface resulting from planing, the specimens were immersed in diluted nitric acid for about three minutes[9] and thereupon annealed in an electric oven for one hour at 240° C. At the end of this time the current was turned off and the specimens remained in the oven over night.

The specimens were then ready for etching in order to detect the strain lines. The process used was that originated by Dr. A. Fry of Essen, Germany.[10] As macrographic examination was desired, solution No. 1 was used. The bath was made up as follows:

240 c.c. HCL (concentrated)
200 c.c. water
90 grams crystallized copper chloride

The specimens were immersed in the bath, which was frequently stirred, and were taken out at regular intervals for examination. Upon removal from the bath the specimens were dried, then washed with alcohol and again dried. After about two hours the strain lines began to appear faintly. At the end of two and one-half hours the bath was changed and the specimens were then left immersed in the bath overnight, a period of about eighteen hours. During this time the bath was neither stirred nor changed, and since the bath contained a weaker solution, the rate of etching after the first two or three hours was probably very slow. The same result might perhaps have been attained more quickly had the bath been stirred and changed at frequent intervals during the night. In the morning the specimens were examined and the bath changed, after which the specimens were allowed to remain in the bath about one hour longer in order to remove a few spots of copper which had deposited on the surface overnight.

[9] Essentially as recommended by J. Dudley Jevons, Journ. of Iron & Steel Inst., Vol. CXI, pp. 191 *et seq.*

[10] Stahl und Eisen, Vol. XLI, pp. 1093.

The strain lines now appeared very distinctly, as shown on Fig. 27, where the two specimens are photographed together. The lines show as dark bands which appear continuous from the outside surface on down across the middle surface, where they are somewhat thinner.

FIG. 27
Lüders' Lines in two specimens of pipe-wall cut from the 14-in. pipe planed down to 40 per cent of the original thickness and etched

It is clear that the lines are not merely a surface phenomenon. It seems probable that they are primarily associated with the longitudinal stresses, which alone persist throughout the wall

thickness, and which therefore appear to determine the strength of the pipe bend as a whole.

E. The Strength of Pipe Bends

When in the present case we speak of the strength of pipe bends, we have in mind their elastic power of resistance. The object of pipe bends is to provide sufficient elastic yielding capacity in a pipe line so that the required rise of temperature shall not cause excessive strains or reactions anywhere in the system. In practical engineering we want to be able to predict whether a designed lay-out of piping with given pipe bends will be satisfactory and reliable and hence we need a criterion for the limit of strength, that is, for the states of stress at which a pipe bend will begin to suffer a permanent set. This criterion should be based on a comparison between an easily calculable function of the stresses and a readily obtainable physical constant of the material, such as the elastic limit or the yield point.

We are thus led to consider the *Theory of Strength* of materials under combined stress, the aim being to determine the permissible stress, actual or "equivalent," inside of which failure will not occur. We are here particularly interested in ductile materials, where failure is always preceded by or associated with plastic flow. The question becomes then to determine the limiting states of stress at which plastic flow sets in, or, as we may say, the *limit of plasticity*. We shall here discuss only the more modern theories and merely mention Rankine's *maximum stress theory* and Saint-Venant's *maximum strain theory*, both of which have been shown to be incomplete.

The theory which in recent years has found most general acceptance is the *maximum shear theory* proposed by Coulomb and Tresca and supported by experiments of Guest and others. According to this theory failure of ductile materials, such as low carbon steel and copper, is due to a permanent displacement or sliding in the plane of maximum shear. In the plane of sliding the material flows and the hypothesis is that flow will not occur until the shear stress reaches a certain limit. According to Professor Otto Mohr and as already found by Guest, the limit of plasticity is independent of the middle principal stress, and is determined entirely

by the algebraically greatest and smallest stresses, say p_1 and p_2. The criterion of strength is simply the value of the maximum shear stress given above:

$$q_3 = \pm \tfrac{1}{2} (p_1 - p_2) \tag{28}$$

and when this reaches a certain constant value, whatever may be the state of stress, flow will occur. This value can be readily determined for simple one-axial tension, such as occurs in ordinary tensional tests where p_2 is zero and $q_{max} = p_1/2$.

Denoting by p_0 the value of p_1 at the yield point in simple tension, we have

$$q_{max} = \frac{p_1}{2} = \frac{p_0}{2}. \tag{29}$$

Hence, in general, substituting in (28) we find the condition of failure according to this theory:

$$p_1 - p_2 = p_0. \tag{30}$$

As mentioned in Chapter A, A. M. Wahl and J. R. Finniecome propose to use the maximum shear theory in the design of pipe bends, and also the author of this paper has made attempts in the same direction; but recent researches have shown that the middle principal stress is not without influence on strength. In fact, in certain states of stress the maximum shear theory may be in error by as much as 15 per cent. Another and more comprehensive theory is needed to account for all the facts.

Such a theory has been framed by Dr. M. T. Huber and Dr. H. Hencky, leading to an equation for the condition of plasticity. Professor Dr. R. v. Mises has arrived at the same result although along a different line of reasoning.[11] As advanced by Huber-Hencky the theory is based on the hypothesis that the limit at which plastic flow begins, depends on the elastic work of deformation. When this work attains a certain constant value at a given point, regardless of the distribution of stresses by which it is produced, the limit of plasticity will be reached at that point. By

[11] M. T. Huber, "Die spezifische Formänderungsarbeit als Mass der Anstrengung eines Materials." Lemberg 1904. See also Föppl, "Drang und Zwang," I, page 50, 1920.
R. von Mises, "Göttinger Nachrichten," 1913, pages 582–592.
H. Hencky: Zeitschr. f. angew. Math. und Mech., Bd. 4, Heft 4, 1924.

work of deformation is here meant the potential energy stored in the material by pure change of form, exclusive of that due to change of vol me. This theory seems very plausible, because in it both stresses and strains are taken into account; it has recently received excellent confirmation by tests made by Professor M. Roš and A. Eichinger in Zürich and by Dr. W. Lode in Göttingen.[12]

The work of pure elastic deformation is given by the expression:

$$W=\frac{m+1}{3mE}\,(p_1^2+p_2^2+p_3^2-p_1p_2-p_2p_3-p_1p_3). \qquad (31)$$

If this is to be constant, the term inside brackets must have a constant value which can be found by equating it to the work in simple one-axial tension:

$$W=\frac{m+1}{3mE}\,p_1^2. \qquad (32)$$

Denoting again by p_0 the known value of p_1 in simple tension at the yield point, and equating the expressions on the right side of the equations (31) and (32), we obtain (23).

In bi-axial stress $p_3=0$ and (23) becomes as given above:

$$p_1^2-p_1p_2+p_2^2=p_0^2. \qquad (24)$$

This does not invalidate the theory that failure takes place by sliding or shearing; we have evidence of this in the Lüders' lines, but according to the new theory the shear stress at the limit of plasticity may be greater than $p_0/2$ in certain states of stress. In fact, recent experiments[13] seem to show that in pure shear, such as may occur in torsion of a tube, the limiting shear stress is equal to about .60 of the stress at the yield point in simple tension or compression of ductile steel. Theoretically the limiting value should be .577 p_0.

The theory of Huber-Henckey is used in the present analysis where, as stated above, the square root of the expression on the

[12] M. Roš and A. Eichinger, "Versuche zur Klärung der Bruchgefahr." Second Intern. Congress f. Tech. Mech., Zürich, 1926.
W. Lode, Forschungsarbeiten a. d. Gebiete des Ingenieurwesens, Hft. 303, 1928.

[13] F. B. Seeley and W. J. Putnam, Uni. of Ill., Eng. Experiment Station, Bull. 115, 1919.

left side of (24) is denoted by p_e and is called the "equivalent stress."

It will be noticed that where the principal stresses are of different sign, p_e will always be greater than p_1, the greatest of the stresses. This occurs in pipe bends, where the transverse stress p_2 on any polar coördinate of a section occurs with both positive and negative sign, being always of opposite sign on the inner and outer surface. Thus the plastic limit may be reached long before either p_1 or p_2 has reached the yield point.

It has been pointed out in the analysis in Chapter D that so long as the bends were not strained beyond the limit of proportionality, the calculated maximum longitudinal stresses at the outer surface, although generally well above 20,000 lbs. per sq. in., did not exceed the elastic limit, while the calculated transverse stresses, as also the equivalent stresses were far beyond that limit in certain regions. In fact, it appears that failure did not occur until the longitudinal stress reached the elastic limit at some part of the pipe. This indicates, as suggested by the study of Lüders' lines, that the strength of pipe bends depends essentially on the longitudinal stresses, a view which is supported also by a study of the peculiar state of stress which exists in the wall of such pipes.

As seen from the curves for equivalent stress in Figs. 13, 16 and 19, the most highly stressed parts, where flow of the material is likely to begin, are found in the outer surface in four bands, at first of small width and probably of very small depth. One of these bands is located on each side of the section near the level of the axis, one at the top of the section and one at the bottom. We have similarly four bands of high equivalent stresses on the inner pipe wall. In the middle surface, where the transverse stress is zero, the longitudinal stress prevails alone, but as we go radially outwards or inwards, the transverse stresses increase to maxima at the outer and inner surfaces, while the longitudinal stress varies relatively little and is in fact the same throughout the whole thickness in the four points where $\theta = 45°$. The zones of plasticity, although at first strictly local, exist at the points where the transverse and hence the equivalent stresses are greatest, and will enhance transverse bending and flattening of the section. They will not, on the other hand, have any direct influence on the bend-

ing of the pipe as a whole, which is resisted only by the longitudinal stresses, and these stresses are relatively small in the zones of plasticity. It may thus be explained that the elastic limit is passed in certain places although the calculated longitudinal stresses are still moderate and the pipe as a whole does not show an appreciable permanent deflection. Gradually as the bending moment is increased, plasticity will spread, and greater stresses will be thrown on adjacent material until an elastic-plastic equilibrium is established. When now the load is taken off, the potential energy in the elastic parts of the pipe will be released and tend to bring the bend back to its original shape, but the metal which has suffered plastic flow will not return quite to its original state, the restitution will not be complete and a permanent set of the bend will result. When the load is re-applied and the bending moment still further increased, plasticity will begin to spread again, and flattening of the section will progress, making the stress distribution more and more unfavorable, until finally a point is reached where the stress couple cannot further increase. If now the load persists, as it does in the testing machine, the material will continue to flow, the stress couple and hence the load falls off, and the pipe bend breaks down. During this process the lower part of the section, becoming more and more flat and being subject to compression, will ultimately yield by an upward buckling or caving in. This actually happened, especially in the 14-in. pipe, and, as stated above, must have caused the neutral axis to move upwards.

It appears then that in the three pipe bends here under consideration *the strength was determined chiefly by the magnitude and distribution of the longitudinal stresses, in particular those in the middle surface*, but it would not be safe to conclude without further tests that such will always be the case. In this connection it is of interest to place the tests here recorded in their true perspective by tabulating and comparing the values of Karman's coefficient λ and the coefficient K where

$$\lambda = \frac{2hR}{r^2} \text{ and } K = \frac{48h^2R^2+10r^4}{48h^2R^2+r^4} = \frac{12\lambda^2+10}{12\lambda^2+1}.$$

TABLE 5

Diameter of Pipe Bend	λ	K
6 in.	.785	2.07
9½ in.	.509	3.19
14 in.	.662	2.44

It is seen from this table that the tests deal with a rather limited range of dimensions, although they fall in a region of pipe bends not uncommonly used in practice when steam pressures are moderate. Where high pressures are used, the wall thickness $2h$ is greater, λ is greater, and then K approaches unity. Tests on such pipes would be of great interest.

F. Summary and Conclusions

The tests here recorded and analyzed confirm the correctness of the formulas for deflections and stresses in pipe bends developed in the 1926 paper, in which account is taken of the deformation by flattening of the section of a curved pipe subject to bending.

In the present paper more complete and accurate formulas are given for calculation of the stresses than those proposed in the 1926 paper, where the longitudinal and transverse stresses were considered to be independent of each other. Actually longitudinal stresses exist in the pure form only at the middle surface of the pipe wall. At the outer and inner surfaces transverse stresses exist at the same time and for a complete solution it is necessary to take into account the interaction of both, calculating each of them from the longitudinal and transverse strains with due regard to Poisson's ratio. This is done by the formulas (7).

The main purpose of the tests here described is to find a simple criterion for the strength of pipe bends, or rather to present experimental material and analysis which will be of assistance in solving this problem.

The analysis showed that long before a permanent set occurred in the pipe bends, the calculated transverse stresses as also the equivalent stresses, calculated by the Huber-Hencky method, exceeded the yield point in the section at the top of the bends, both on the sides of the section near the level of the axis and at its top and bottom. Plastic flow must have taken place locally

in these places. Also the calculated principal shear stress at the outer surface exceeded the limit at which sliding should take place according to the maximum shear theory, before the pipe took a permanent set. It is clear then that none of those stresses can be used as a criterion of strength in new design.

The calculated longitudinal stresses in the outer surface, on the other hand, seemed to reach the elastic limit at about the same time that a measurable permanent set began to appear, while the stresses at the middle surface were well below that limit. Now, the longitudinal stresses exist throughout the wall thickness of the pipe with only moderate variation, reaching extremes at the outer and inner surface. They constitute the stress couple which directly resists the bending moment and since they did not reach the yield point before the pipe as a whole took a permanent set, it seems logical to use them as the criterion of strength.

The observed longitudinal stresses, as derived from direct measurements of the strains were indeed much greater than the calculated stresses as soon as the latter approached the elastic limit, but the picture of the stress distribution so obtained cannot be correct, since the stress couples computed from the "observed" curves is far in excess of the bending moment. At the places where the equivalent stress exceeds the yield point there is plasticity, and the stresses do not reach the values corresponding to the strains. The flattening of the section increases and a gradual redistribution of the stresses takes place. The discrepancy between observed and calculated stresses simply indicates a local flow of the material, which does not at once influence the behavior of the pipe.

A further explanation of this discrepancy may be sought in irregularities in the thickness of the pipe wall consequent upon manufacture.

A study was made of the lines which appeared on the surface of the pipes soon after a permanent set had occurred, the so-called strain lines or Lüders' lines, and it was attempted to correlate them with the various stresses as shown in Figs. 25 and 26. No correlation seemed to exist either in case of the transverse stresses (maximum stress theory), the shear stresses (maximum shear theory), or the equivalent stresses (Huber-Hencky theory). Only

in case of the longitudinal stresses was it possible to establish a correlation. In fact, a fair correspondence in the location of the Lüders' lines and the maxima of the curves for longitudinal stresses existed, especially for those at the middle surface. The maximum points of the curves fell indeed somewhat lower than the lines, but this discrepancy may perhaps be explained by the fact that the lines did not develop fully until after a permanent set had occurred, when the pipe had commenced to flatten and cave in on the under side. When this happened, the neutral axis moved upwards, whereby the regions of maximum longitudinal stresses would likewise move upward and might well occur in the same locations as the strain lines.

Accepting as a tentative hypothesis the theory that the longitudinal stresses and in particular those in the middle surface, determine the point of breakdown of pipe bends, it becomes of interest in design work to calculate the maximum longitudinal stress in a given bend subject to certain assumed deflections or end forces. The longitudinal stress at the outer surface can be calculated from formula (20) for any assumed principal dimensions and bending moment, and its maximum values can be found by means of (20a). The stress at the middle surface can be calculated from (20) simply omitting the last term inside brackets, whereby we obtain the same expression as given in the 1926 paper:

$$p_1 = \frac{KMr}{I}\left(\sin\theta - \frac{6r^4 \sin^3\theta}{24h^2R^2+5r^4}\right). \qquad (20')$$

Its maximum value is found from the simple formula:

$$p_{1\max} = \pm\frac{2}{3}\frac{KM}{rI}\sqrt{\frac{24h^2R^2+5r^4}{18}} \qquad (21)$$

already given in Chapter B.

It would appear from the tests here analyzed that so long as the stress determined by (21) does not exceed about 20,000 lbs. per sq. in. there will be no permanent set in the pipe bend. Should a rule of this form prove applicable also to bends of other types and dimensions, it would afford a fairly simple criterion for the strength of pipe bends. The calculation could be facilitated by

tables and diagrams. But before such generalization can be safely made, it must be verified by further experiments and by an extensive study of the behavior of pipe lines in actual service. Tests on high-pressure steam pipes, when λ is greater and K is smaller than in the tests here described, would be of particular interest.

Acknowledgment is due to Prof. E. F. Miller, head of the Department of Mechanical Engineering, for permission to use the Testing Materials Laboratory of the Institute and to Prof. H. W. Hayward and Prof. I. H. Cowdry of the same department for advice and assistance in carrying out the tests.

The tests of the early part of 1928 were made by Lieuts. H. A. Schade, E. E. Sprung, G. A. Holderness and J. J. Herlihy in preparation for their thesis, presented June, 1928. The tests of October, 1928, were made by Lieuts. A. M. Morgan and R. S. Hatcher in preparation for a thesis which is intended to comprise further tests and is to be presented in June, 1929. The photographs here reproduced were taken in connection with the above mentioned 1928 thesis, from which is abstracted also the description of tests with the 9½-in. and 14-in. pipe bends as well as the investigation of Lüders' lines by etching. Finally the strain diagrams, Figs. 10 and 11, are taken from that thesis.

ON THE METHODS OF CLASSICAL AND QUANTUM MECHANICS

By N. H. Frank

Introduction

There is a widespread feeling among physicists (except perhaps those directly engaged in the problems of atomic physics) that there is an ever-widening gap between the methods of classical and quantum mechanics. Many of these scientists lament the fact that the newer developments have lost the chief charm of the classical method, *viz.*, the simple and direct appeal to naive intuition by its "reasonable" concepts, and that a formalism is creeping into the foundations of the newer methods which was quite lacking in the classical viewpoint. It is the purpose of this paper to show that the philosophical basis of quantum mechanics rests on no less secure or arbitrary foundation than Newtonian mechanics.

1. Classical Mechanics.

Before we discuss the direct comparison, let us first, by way of preparation, glance back at some of the general principles of the scientific method. It is characteristic of all scientific method to form conceptual models of external things in nature, and to use these models to help correlate and predict experience. The only restriction on these models is, that if they have a one-to-one correspondence with nature, then the logical consequents of these models must themselves have a one-to-one correspondence with the subsequent state of nature. It is not possible, *a priori*, to ascertain that such models can be found, and, indeed, we believe that such correspondence exists only because of the past success of this scientific mode of procedure.

Often, when talking about the same things, we employ quite different models. As a simple example, consider a steel projectile; for purposes of determining its trajectory, we think of it as a material point; when we study its behavior in the gun, we con-

ceive it as a rigid body rotating about its axis; when studying some of its tensile properties, the model becomes an elastic continuum and when investigating others, we consider it a space lattice structure of molecules. Each of these models is of use in its particular range of application, and each is equally "true." Concepts such as these, we shall call models of the perceptual world.

There is another type of concept which is introduced in science and which is of quite a different nature than the model. These are the concepts which are employed to help us deduce the relations between models and predict how the models will behave. In physics, such concepts as force, energy and entropy fall into this category. No objective reality, in the sense of correspondence to part of the perceptual world, can be given these concepts. In the following we shall speak of the models as "A" concepts, and of the latter type as "B" concepts.

Let us now turn to Newtonian mechanics, and examine some of those "A" and "B" concepts which are usually considered "reasonable." It will suffice, for the present purpose, to examine one of the simplest "A" concepts (*i.e.*, working model) in this field, the material point. For the time being, we shall accept the concept of Euclidian, three dimensional space and independent time as purely intuitive. The term matter shall refer to the classification of certain of our sense-impressions. How or why this classification is made need not concern us here. In studying the motion of a material point it is treated as a geometrical point, but it is necessary to give some criterion as to when this concept is applicable to experience. For this purpose, we define a material, or mass point as any body whose dimensions are vanishingly small compared to *all* the measurements necessary to specify its motion. By the position of a mass point, we can only mean the specification of the position of one of the geometrical points which lies within the space extension of the material point. There will obviously be an infinite number of these. Even in such a simple concept as this, we find a basic uncertainty in the definition of an observable physical quantity. The "exact" laws of mechanics are exact only because we express these laws in terms of mathematical concepts (in our case, geometrical point), and when interpreted

physically do not allow exact significance. In thus determining the position of a mass point, which of the geometrical points shall we use? If we say, as we must, *any* one falling within the finite volume defined by the mass point, it follows that an exact number (the coördinate of a geometrical point), when interpreted physically, means any number within a certain limited set.

The same limitation is imposed on the concepts of velocity and acceleration. The differential coefficients used to define these quantities must also be interpreted as any number in a limited set. All numbers within such sets are physically indistinguishable. How do we know that such a model and such concepts will work in correlating and describing experience? We do *not* know until we have submitted them to the test of experiment; only then have they a utility, *i.e.*, are they "so.". In the range of macroscopic, or large scale phenomena, the limitations of precision of measurements are such that it is never necessary to reduce our mass points to atomic dimensions to have them satisfy our definition. Thus, as long as we confine ourselves to measurements of such an order of magnitude that, within the utmost precision obtainable, we can choose our mass points with dimensions very much greater than those of molecules (and still be mass points in our sense), Newtonian mechanics forms a model which has been remarkably successful in describing and correlating a tremendous amount of observable phenomena.

2. **Quantum Theory.**

When it became possible to carry out experimental investigations involving determinations of lengths of the order of magnitude of atomic dimensions, it was very natural and rational to attempt to use the classical model in describing these new experiences. Is it possible, *a priori*, to predict the success of such procedure? Again, just as in the case of macroscopic phenomena, we do not know until we have tried. Just because the ideas have been useful in one range of experience is no guarantee, nor even sufficient reason to suppose, that they would be applicable in any other range.

Upon trial it is found that we cannot predict experience, as we have mentioned, with this time-honored model. Even though

we have become very much impressed by the success of the classical method in its region of applicability, we should feel no hesitation in denying its validity in this range of experience. Not to do so is an indication of the fact that the assumptions and appeal to experience (upon which any physical science must be based) have been mistaken for mystical truths, or that they have never been clearly realized.

Granted, then, that the old ideas must be modified, how? In formulating a definite answer, let us examine a simple example, to see how the attempt to carry over classical mechanics has proceeded. In the investigation of atomic phenomena, physics has labeled one type of experience the electron. Proceeding according to classical methods, it forms a model of the electron as a mass point, with all the attendant restrictions demanded by its definition. An important difference is at once noticed in the fact that now we cannot directly determine the coördinates or momenta, but must get at them indirectly. The attempts to use this notion, as we have already stated, lead to definite contradictions of experience.

As is well known, Planck was the first to successfully depart from classical models, although the mode of departure was only implicitly contained in his assumptions. He pointed out that it was apparently not possible to consider the electron (or more generally the molecule) as a mass point, because an inherent uncertainty in the determination of the coördinates and momenta of the molecules always was present, and that it was *much larger* than the corresponding dimensions of the mass point model which was employed. To bring out this idea more clearly, let us consider a mass point of one degree of freedom. We say that the mechanical state of this system is determined when the coördinate q and the momentum p are determined. The experimental determination of q involves an uncertainty Δq and the determination of p involves an uncertainty Δp. If Δq is greater (but of the same order of magnitude) than the linear dimensions of the mass point and Δp is correspondingly large, classical mechanics is a satisfactory means of describing this system. It is often convenient to represent this system by a geometrical point in a plane (phase space for one degree of freedom) which is located by two Cartesian

coördinates p and q. We can imagine the plane divided into a network of rectangular cells. If we use cells of area $\Delta_1 p \Delta_1 q$, where $\Delta_1 p < \Delta p$ and $\Delta_1 q < \Delta q$ then these cells form a physical continuum, as neighboring cells are mutually indistinguishable (physically, of course), and integration leads to physically interpretable results.

In this picture, the essence of Planck's assumption is as follows: it is not possible to utilize cells in phase space (in the case of one degree of freedom) which are smaller than a quantity h ($\Delta_1 p \Delta_1 q = h$). This means, that if molecules (or electrons) are to be considered as mass points, there is a lower limit to the size we can choose; and when we restrict ourselves in this way, our mass point is not small enough to satisfy our original definition. Although all we have stated is implicitly contained in Planck's original work, a quarter of a century elapsed before Heisenberg presented the assumptions as a philosophical, rather than a heuristic principle in physics: the uncertainty principle. Considering the electron, for example, as a mass point, this principle states that it is impossible by any conceivable physical experiment to reduce the uncertainty in the determination of the coördinate and the momentum below the point where the product of these uncertainties is of the order of magnitude of h.

As physics progressed, it became clear that Bohr's application of the quantum rule to atom-models, which involved the postulate of classical stationary states and non-classical transitions between these states, was quite insufficient to describe more than a limited range of atomic experience, in which range, however, it did work with remarkable success. The developments of modern quantum mechanics consist in the main of trying to find the best way in which to supplant the classical model by a new one and to set up such a model. These attempts may be divided into two groups; first, the matrix mechanics due to Heisenberg, Born and their associates, and developed in a slightly different manner by Dirac, and secondly, the wave mechanics of de Broglie and Schrödinger.

Let us first consider the common characteristics of both these attempts. Their common basis consists of adhering to the form of the differential equations of classical mechanics. Just what is

retained in so doing? It seems that the correct answer to this question is the following: there is retained that part of the foundations of classical mechanics which states that it is simplest to consider the state of a body determined by the positions and velocities of neighboring bodies. In other words, differential equations of the *second* order are still to be used in describing atomic phenomena.

In the matrix development, the underlying idea has been to retain further the concept of mass point, but to discard the possibility of specifying its position or velocity in terms of the real numbers which are usually interpreted as physical measures of these quantities. Heisenberg has tried to solve the differential equations, not with real numbers, but with matrices in order to get physically interpretable results. Dirac's efforts have led to the introduction of a new kind of number, the so-called q-number, which is equivalent to matrices as far as the needs of the new formulations demand, but need not be identified with them. So, to summarize, matrix mechanics attempts to redefine the operations which must be performed on the differential equations of classical mechanics to get relations which could be successfully used in obtaining physical results.

In wave mechanics, no attempt is made to reject the use of real numbers in solving the differential equations, or to reinterpret the physical meanings which are usually given to these symbols. The departure is made in denying the applicability of the mass point concept. If this is done, what shall be used to replace it? It has been found that it is satisfactory to replace the mass point model by a model which, because it appears as a solution of certain classical equations known as wave equations, is called a wave model. The properties of this model, other than its wave properties which result from the defining equations, are as yet undetermined. The fact that it has been shown that both this and the matrix attempt are mathematically equivalent must not be confused with the difference in attitude and philosophic basis of the two theories. It is indeed not surprising that two theories which arrive at the same symbolic relations should have a mathematical, but not necessarily physical equivalence.

Thus, we see that the efforts of physics today fall in line with,

rather than depart from, the mode of attack which has always been man's tool in trying to build a world-picture of his experience.

Summary

Let us now briefly tabulate the main points of the discussion:

1. Classical mechanics is based on certain assumptions and concepts which are justifiable only in the light of experience.

2. These assumptions and concepts are inadequate to describe experience in the realm of atomic phenomena.

3. To form a new model in this region, physics is attempting to retain as many of these older assumptions and concepts as is possible.

4. These attempts fall into two classes, matrix and wave mechanics; characterized as follows:

(*a*) Matrix mechanics discards only the concepts associated with the determination of physical quantities by real number solutions of the classical equations.

(*b*) Wave mechanics discards only the possibility of the mass point concept as a useful aid in correlating and predicting certain atomic experience.

A NEW MACHINE FOR INTEGRATING A FUNCTIONAL PRODUCT

By KING E. GOULD

Introduction. This paper is intended to describe a new machine, conceived by Dr. Wiener, which performs certain integrations which are important in engineering, and to outline some of the solutions of mathematical problems which these integrations make possible. The two types of integrals which this machine may be made to evaluate and which are of principal importance are $\int_0^t A(\lambda)E(t-\lambda)d\lambda$, which is the essential part of what is commonly known in electrical engineering as Carson's Extension Formula, and $\int_A^B f(\lambda)\begin{matrix}\sin\\ \cos\end{matrix}(n\lambda)d\lambda$, which is to be plotted as a function of n, where A and B are any fixed limits. Where f converges with sufficient rapidity, the evaluation of the latter integral permits of the approximate evaluation of $\int_0^\infty f(\lambda)\begin{matrix}\sin\\ \cos\end{matrix}(n\lambda)d\lambda$. The first ntegral is important because it connects the indicial admittance, or current which will flow when a constant uni-directional voltage is suddenly applied to a linear circuit, with the resultant current which will flow when a voltage of any known character or wave form is suddenly applied to the same circuit. The last integral constitutes an exceedingly powerful tool; it would make possible, for instance, the solution of any linear differential equation or partial differential equation with coefficients which are not functions of the independent variable. Practically the only restrictions to be placed upon the functions A and E in the first integral, and upon f in the second, is that they may be plotted upon rectangular cross-section paper, which means that they must not have too many discontinuities nor become too large. The device for performing these integrations, although not intended for highly accurate work, should eventually have an accuracy high enough for most engineering purposes.

Basic Idea. The basic idea of this machine is illustrated in Fig. 1. Let R–R be imagined as a plane, rectangular surface of height h and length $2m$, from which there is some sort of radiation, say infra-red rays due to raising the temperature of the surface R–R above that of the surrounding air by perhaps 100° C. Suppose that the radiation per unit area is constant over the surface R–R. Let S' be a screen which sets very close to R–R, and of the same size as R–R, the material from which S' is made being completely opaque to the radiation from R–R. At a distance l from R–R and S', S'' is placed as shown, parallel to R–R and S', and symmetrical with respect to the axis A–A, S'' being of the same material as S', and of height h and length m. Let T be imagined as a very narrow strip lying in the plane B–B and perpendicular to A–A, of width 2 $(\Delta^2 T)$ and length $3h$.

Suppose that an aperture is cut from screen S', with a height

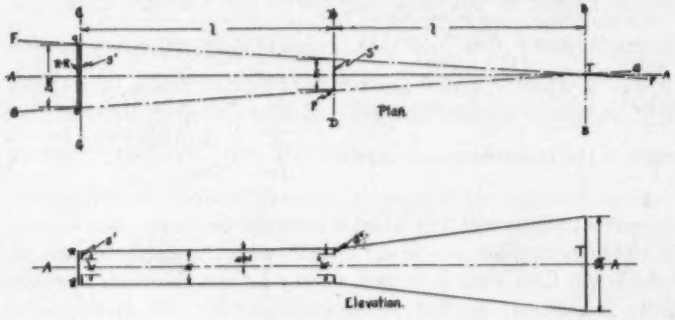


Fig 1.

at each point along the axis C–C given by $f(x)$, where x is measured from O. An aperture is likewise cut out of screen S'', with a height at each point a distance x from p given by $\phi(2x)$, where x is measured in the same units along C–C and D–D. Let us call the height of the aperture in S' at A–A, K', and the height of the aperture in S' at A–A, K''.

Consider the narrow wedge formed by passing two vertical planes, P' and P'' (not shown in Fig. 1) through T, and intersecting C–C at distances Δx from A–A, one plane on one side of

A–A and the second plane on the other side of A–A between R–R and T. Let the intersections of these planes with B–B be a distance $\Delta^2 x$ from A–A, where $\Delta^2 x$ is small compared with Δx, which in turn is small compared with l.

The radiation from R–R which lies between the two hypothetical planes P' and P'' will be proportional to Δx and to K', as the radiation from R–R has been assumed uniform. If S'' is far enough from S' (that is, if l is large compared with h), the distribution of intensity of radiation falling on the portion of S'' between the planes P' and P'' and of length $\frac{h}{2}$ above and below A–A will be uniform and proportional to Δx and to K'. Of the energy falling on this portion of S'' of width Δx and height h, a fractional part will pass through S'', between the planes P' and P'', proportional to K''. As T has a length $3h$, its midpoint being at A–A, all of the energy which passes through the portion of S'' between P' and P'' will fall on T. Thus the energy incident upon T from the strip of R–R included between P' and P'' is proportional to Δx and to the product of K' and K''. The total energy, then, incident upon T from the entire surface of R–R will be the sum of all the energies coming from R–R between closely spaced planes such as P' and P'' included between the planes F–F and G–G, or the total energy incident upon T will be proportional to the integral of the product of the ordinates of f and ϕ between the limits O and $2m$ for f, and O and m for ϕ, provided m is small compared with l, so that R–R is practically the same distance from T for the wedge cut by any two planes such as P' and P''. If W represents the energy incident upon T, then

$$W \propto \int_0^{2m} f(x)\phi(2x)dx.$$

Let us recall the assumptions which have been made concerning the dimensions of the machine. They are:

1. h is small compared with l. This restriction is necessary if the distribution of the intensity of radiation incident upon the vertical elements of S'' is to be uniform; otherwise the energy which passes through S'' between P' and P'' is not proportional to the ordinate at S''.

2. m is small compared with l. This restriction is necessary if all the vertical elements of R–R are to produce the same increase of energy incident upon T, assuming that the product of the ordinates at every value of x for ϕ, and $2x$ for f, are constant.

3. The width of T is small compared with m. This is necessary if the correct ordinates of f and ϕ are to be multiplied; otherwise the multiplication is not limited to ordinates which correspond to x for f twice the values of x for ϕ.

If, now, S' is turned through an angle of 180°, about a vertical axis through O, (S' extending as a solid sheet for all negative values of x) and displaced with $x=0$ at a distance t from the point O, then

$$W \propto \int_0^t f(t-x)\phi(2x)dx$$

which is of exactly the same form as $\int_0^t A(\lambda)E(t-\lambda)d\lambda$; all that is necessary is to plot ϕ to the correct scale on S''.

If, instead of revolving S', it is simply displaced in the direction of decreasing values of x by a distance t, S' being made to extend along the direction of increasing values of x and $f(x)$ being cut out of S' for values of x greater than $2m$, then

$$W \propto \int_t^{t+2m} f(x+t)\phi(2x)dx,$$

or the limits can be adjusted to give

$$W \propto \int_A^B f(x+t)\phi(2x)dx.$$

Although this is not one of the integrals for which the machine was principally devised, still, it is included in those which the machine will evaluate.

If, with S' in its original position, S'' is moved along A–A, always remaining perpendicular to A–A and with the point P always along the same horizontal line lying in the plane G–G, the scales of both the ordinates and the abscissae of ϕ are multiplied by the same factor, and

$$W \propto n\int_0^{2m} f(x)\phi(2nx)dx$$

so that by moving S'' in the proper manner and with the proper limits, and with the correct scales for x on S'', the energy on T may be made to indicate

$$n\int_A^B f(x)\phi(nx)dx$$

over a range of n limited in its maximum value by how far S'' can be moved toward S' without violating the restriction that h be small compared with the distance between the two screens, and in its minimum value by the restriction that the width of T be small compared with the width of S'' between the planes G–G and F–F.

The expression $\int_A^B f(x)\,\phi\,(nx)\,dx$ includes, of course,

$$\int_A^B f(x)\begin{matrix}\sin\\ \cos\end{matrix}(nx)dx$$

and hence this latter integral can be evaluated by sliding the screen S''. However, for this special case, it is sufficient to be able to evaluate integrals of the form

$$\int_a^b f_1(x)f_2(x+t)dx$$

as a function of t. This may be demonstrated as follows:

$$\int_A^B f(x)\sin(nx)dx = \int_A^B f(x)\sin(\epsilon^{\log n+\log x})dx.$$

Let $x=\epsilon^y$.

Then this expression becomes

$$\int_{\log A}^{\log B} f(\epsilon^y)\sin[\epsilon^{\log n+y}]\epsilon^y dy.$$

Now let

$$\sin\epsilon^u=\psi(u)$$

$$\text{and } f(\epsilon^u)(\epsilon^u)=F(u).$$

Then this becomes

$$\int_{\log A}^{\log B} F(y)\,\psi(\log n+y)dy$$

which is of the form

$$\int_a^b f_1(x)f_2(x+t)dx.$$

A similar demonstration applies to $\int_A^B f(x) \cos(nx)dx$.

Thus the Fourier Transform may in general be evaluated approximately by any device which will give $\int_a^b f_1(x)f_2(x+t)dx$, provided that $f(x)$ converges to zero with sufficient rapidity. Practically, however, there are limitations to the functions which may be evaluated in this way, involving the behavior of $\sin(\epsilon^n)$ and $f(\epsilon^n)\epsilon^n$.

Practical Application of the Basic Idea. As far as the physical dimensions of the machine were concerned, the prime requisite was that the screens out of which the functions are cut be large enough that graphical methods be of reasonable engineering accuracy; the middle screen, S'', being the smaller, was the limiting factor in this respect. It was decided that S'' should be 12 by 12 inches, thus allowing both a maximum ordinate and a maximum abscissa of 12 inches. This fixed the length of S' at 24 inches, giving a maximum abscissa of this length. As the accuracy of the ordinates should be the same on each screen, the height of S' was made 12 inches.

For the length between screens and between S'' and T, 8 feet was chosen. Thus a vertical element of the radiator at O or q, and a similar element at $A-A$ are weighted in the ratio $[\cos(\tan^{-1}\frac{1}{16})]^2:1$ or 0.996:1, and the product of small apertures at the bottom of S' and the top of S'', and small apertures at the same height are weighted in the ratio $[\cos(\tan^{-1}\frac{1}{8})]^2:1$ or 0.985:1. The error of this latter ratio may seem large at first sight, but it must be remembered that these are the two most extreme cases and that ordinarily the two ordinates will be multiplied with far greater accuracy.

The length of the receiver was made 36 inches to allow for maximum ordinates of 12 inches, and the width of the receiver was set at $\frac{1}{8}$ inch. With this width receiver, each ordinate of S' is multi-

plied by approximately the average ordinate over a width of $\frac{1}{16}$ inch in S'', which with ordinary functions gives very nearly the same result as the multiplication of the two functions ordinate by ordinate. With the receiver as actually built, the average ordinate mentioned above becomes a weighted average with heaviest weighting at the axis of the receiver, which decreases any inaccuracy due to the finite width of T.

Infra-red radiation from a black body at a temperature of about 300° C. was used, as it fulfilled best the major requirements that the radiation be capable of production from a plane surface, with uniform intensity over that surface, in quantities large enough to be measurable, that it be unable to pass through material suitable for making the screens, and that it be of such a nature that an instrument could be constructed to measure the total radiation falling on a long, narrow strip. To measure the total radiation falling on the $\frac{1}{8}$ by 36 inch receiver, 256 thermocouples of approximately the same sensitivity were placed at equal intervals along this 36-inch length, and by connecting these elements in series and measuring the total e.m.f. by means of a sensitive galvanometer, the average value of the radiation falling on the receiver per unit length should be obtained with a high degree of accuracy.

As the distance between R–R and S' does not affect the accuracy of the machine as long as R–R is made large enough so that a

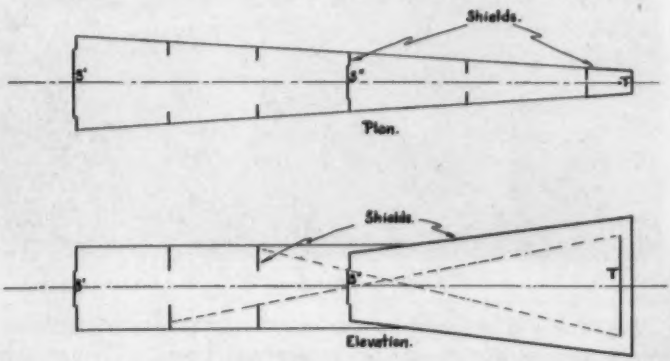


FIG. 2.

straight line from any point on S'' through any point on S' intersects R–R, the radiator was placed a few inches from the screen just in front of it.

The air between the radiator and the thermopile must necessarily absorb some part of the radiation, but as long as the per cent absorption remains constant, this will not affect the accuracy of the device.

Because the thermopile is sensitive to radiation from surrounding objects, it must be shielded from them. To accomplish this, shields and baffles at constant temperature were placed as shown in Fig. 2, so that at no point on T is there received any radiation except from S' or S'' and the baffles and shields. The dotted lines in Fig. 2 illustrate this statement. The screen S' was made to slide horizontally along its length and eventually S'' may be made to move along the length of the machine.

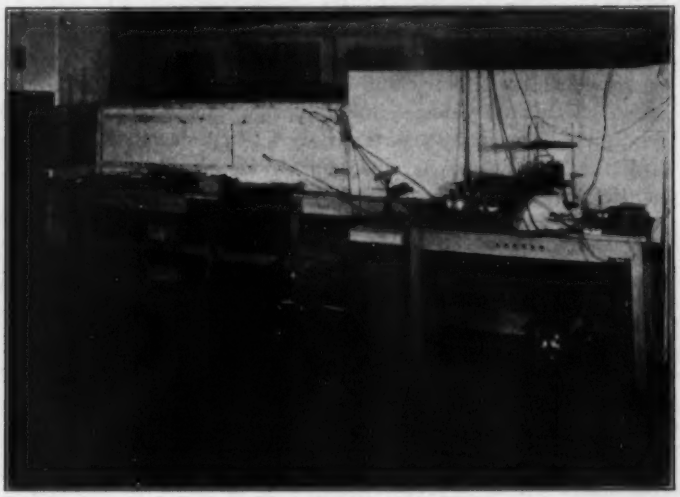


FIG. 3

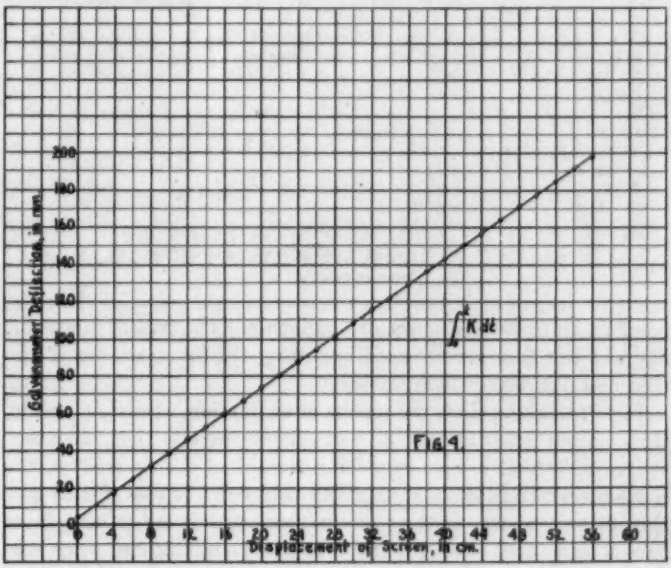
Fig. 3 is a photograph of the machine as it was constructed, giving some idea of its size and general appearance. The radiator is shown at the left of the photograph, against the concrete

column. The screen near the radiator is carried on a frame which slides on a guide, both of which can be seen projecting from the machine near the radiator. The middle screen is placed in a stationary frame which is inside the beaver-board walls, access to this screen being by way of a lid at the middle of the machine. The thermopile, in its case, is mounted vertically inside the machine at the extreme right end. The physical difficulties which had to be overcome and the methods and apparatus employed are to be described in another paper.

Results Obtained. The results obtained may be summarized in a very few words, as follows: a device has been constructed which will evaluate practically any integral of the type

$$\int_0^t A(\lambda)E(t-\lambda)d\lambda$$

for given values of t, provided the functions A and E remain finite and have a finite number of discontinuities between o and t. The



time required for the evaluation at any one value of t, after the functions have been properly set up in the machine, is less than one minute, and the error to be expected in the result is generally less than two per cent for ordinary functions. The time required to set up the functions in the machine will vary widely with the functions themselves, but a fair estimate would be the time required to plot the two functions on rectangular cross-section paper, plus

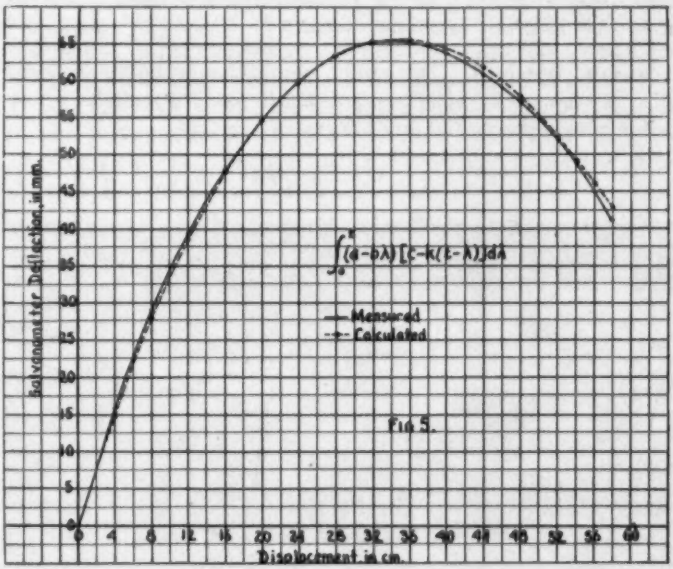
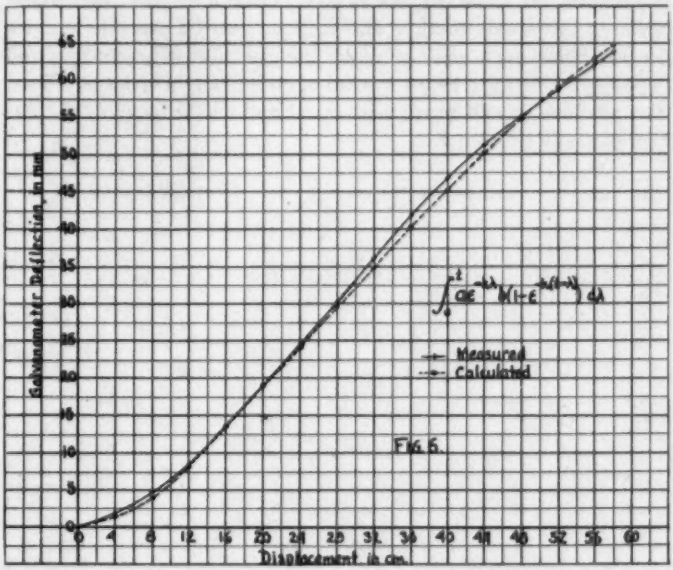


Fig 5.

fifteen minutes to two hours, depending upon the type of functions. Most of this time is consumed in cutting the functions out of the thin metal screens which are used.

Figs. 4, 5 and 6 show the results of evaluating, at several points, three simple integrals, which were chosen because they are easily evaluated formally, thus furnishing a check of the accuracy of the machine, while as far as testing the ability of the machine to give the integral of the product of two functions is concerned, these simple functions served as well as more complicated ones. In

Fig. 4, $\int_0^t K dt$ is plotted as a function of t, K being a constant. This integral was evaluated principally as a test of the uniformity of temperature along the length of the radiator, and it will be seen that all points lie quite close to a straight line. The twenty-eight readings for this curve required about thirty minutes, the



aperture being opened to some value of t, a reading taken and the aperture closed, so that the zero reading might be checked at each point. Figs. 5 and 6 show the results of the measurement, at fifteen values of t, of integrals of the types

$$\int_0^t (a-b\lambda)\,[C-k(t-\lambda)d\lambda \text{ and } \int_0^t a\epsilon^{-k\lambda}[b(1-\epsilon^{-k(t-\lambda)}]d\lambda$$

respectively a, b, c and k being constants although not the same in the two cases. It will be noted that in Figs. 5 and 6 the curves obtained by means of the machine check to quite fair engineering

accuracy with the calculated curves, shown dotted in each case. The machine was calibrated in each case by taking a reading with the apertures open.

Thus there seems to be little doubt that this apparatus has progressed to the point where it is practically applicable to the evaluation of integrals of the type commonly known in electrical engineering as Carson's Extension Formula. Much refinement towards increased stability and accuracy seems possible, the intention being eventually to make the machine self-recording and capable of plotting a result instead of giving only a series of points. The machine should prove a practical and useful tool in the solution of many problems.